NOLOGY DEPT.

THE AMERICAN MATHEMATICAL MONTHLY

THE OFFICIAL JOURNAL OF
THE MATHEMATICAL ASSOCIATION OF AMERICA, INC.

VOLUME 63 NUMBER 9

PUBLIC LIBRARY
NOV 21 1956
DETROIT

PART I

CONTENTS

Campanological Groups. . . . . T. J. FLETCHER 619
Polynomials Defined by Generating Relations . . . . . R. P. BOAS, JR. AND R. C. BUCK 626
A Note on Quasi-Idempotent Matrices . . . . C. R. MARATHE 632
Algebraic Properties of Classes of Functions. . . . . A. A. GRAU AND B. T. GOLDBECK, JR. 636
College Mathematics for Non-science Students 639
Mathematical Notes. JOHN GURLAND, R. J. LEVIT, J. R. HATCHER, ... ALBERT WILANSKY, E. B. LEACH 643
Classroom Notes. . . H. A. ARNOLD, J. H. MEANS, HARRY LEVY, ... E. B. LEACH, PHILIP FRANKLIN 656
Elementary Problems and Solutions 664
Advanced Problems and Solutions 669
Recent Publications 673
News and Notices 677
The Mathematical Association of America 683
Thirty-seventh Summer Meeting 683
New Members. 686
May Meeting of the Wisconsin Section 687
Organizational Meeting of the New Jersey Section 688

NOVEMBER 1956

The AMERICAN MATHEMATICAL MONTHLY

(Founded in 1894 by Benjamin F. Finkel)

Carl B. Allendoerfer, *Editor*

ASSOCIATE EDITORS

R. V. CHURCHILL	EDWIN HEWITT	E. P. STARKE
HOWARD EVES	L. M. KELLY	G. B. THOMAS
F. A. FICKEN	IVAN NIVEN	E. P. VANCE
WALLACE GIVENS	EDITH R. SCHNECKENBURGER	H. S. ZUCKERMAN
	GEORGE SPRINGER	

EDITORIAL CORRESPONDENCE should be addressed to the Editor, C. B. Allendoerfer, Department of Mathematics, University of Washington, Seattle 5, Washington.

ADVERTISING CORRESPONDENCE should be addressed to F. R. Olson, University of Buffalo, Buffalo 14, N. Y.

NOTICE OF CHANGE OF ADDRESS by members of the Association as well as correspondence regarding subscriptions to the Monthly should be sent to the Secretary-Treasurer, H. M. Gehman, University of Buffalo, Buffalo 14, N. Y.

THIS IS THE OFFICIAL JOURNAL OF THE
MATHEMATICAL ASSOCIATION OF AMERICA, INC.
(Devoted to the Interests of Collegiate Mathematics)
OFFICERS OF THE ASSOCIATION

President, W. L. Duren, Jr., University of Virginia
First Vice-President, R. V. Churchill, University of Michigan
Second Vice-President, G. B. Price, University of Kansas
Secretary-Treasurer, H. M. Gehman, University of Buffalo
Associate Secretary, Edith R. Schneckenburger, University of Buffalo
Editor, C. B. Allendoerfer, University of Washington

Additional Members of the Board of Governors: M. A. Basoco, H. W. Brinkmann, P. H. Daus, H. H. Downing, P. D. Edwards, N. J. Fine, R. M. Foster, Philip Franklin, F. F. Helton, M. S. Hendrickson, A. S. Householder, R. C. Huffer, C. A. Hutchinson, L. W. Johnson, G. K. Kalisch, E. C. Kiefer, F. W. Kokomoor, R. E. Langer, Z. L. Loflin, L. L. Lowenstein, Saunders MacLane, E. J. McShane, W. H. Myers, Ivan Niven, Morris Ostrofsky, O. J. Ramler, J. F. Randolph, C. B. Read, C. R. Sherer, M. F. Smiley, C. E. Springer, B. M. Stewart, G. B. Thomas, Jr., Bernard Vinograde, M. A. Zorn

---

Entered as second class matter at the post office at Menasha, Wis. Acceptance for mailing at special rate of postage provided for in the Act of February 28, 1925, embodied in Paragraph 4, Section 538, P. L. and R., authorized April 1, 1926.

Subscription Price: To Members, $4 a Year. To Others, $5 a Year.

Published by the Association at Menasha, Wisconsin, and Seattle, Wash., during the months of January, February, March, April, May, June–July, August–September, October, November, December.

PRINTED IN THE UNITED STATES OF AMERICA

b
n
c
o
r
I
t
t
c
s

CAMPANOLOGICAL GROUPS

T. J. FLETCHER, Sir John Cass College

The art of change ringing has always been considered highly mathematical, but little has been said about the precise mathematical notions involved. This may be because the art has been restricted almost entirely to English-speaking countries, and so has not attracted the attentions of algebraists on the continent of Europe. This paper contains examples of some of the methods used by bell-ringers, and shows that they are decompositions of a symmetric group into Lagrange cosets. In consequence they provide useful practical illustrations of a topic which most textbooks treat in an entirely abstract fashion. We also see that a knowledge of the elementary notions of group theory might have enabled campanologists to construct peals without the tedious empirical methods which seem to have been the practice in the past.

The problem which a team of ringers undertakes is to produce all the permutations (changes) on a set of bells, proceeding according to certain rules. The highest bell, number one, is called the *Treble*, and the lowest is called the *Tenor*. When the bells ring down the scale, from Treble to Tenor, they are said to be in *rounds*. The rules to which a peal must conform are:

i) the peal must begin and end in rounds,

ii) no bell may move more than one place between successive changes,

iii) no bell may lie still for more than two successive changes.

(The last rule is relaxed occasionally.)

A peal may be composed of any number of bells up to twelve, and as it takes about 28 hours to ring all the changes on eight bells alone, the ringers cannot hope to ring all the possible changes if the number of bells is larger. Then they restrict attention to ringing those changes which are regarded as the most musical.

The six changes on three bells may be rung as follows:

1 2 3
2 1 3
2 3 1
3 2 1
3 1 2
1 3 2
1 2 3

They could equally well be rung in the reverse order, but only these two ways conform to the rules. The course of the Treble is particularly simple—it *hunts up* and then *hunts down*. The other bells follow a similar *hunting course* but naturally the phase is different.

Notation. We will denote changes as ringers do by numbering the bells from Treble to Tenor and giving the order in which they ring in the change. The symbols which result could be regarded as the elements of a symmetric group, but it is more strictly logical to regard the transitions between changes as constituting the group, and we will denote these by the familiar cycle notation. Thus the transition

$$\begin{bmatrix} 213654 \\ 514263 \end{bmatrix}$$

will be denoted by the symbol (145)(36). (Rule ii of course prohibits the carrying out of this transition in one move.) The notation means that the bell in first place moves to fourth place, and the bell in fourth place moves to fifth place, and the bell in fifth place moves to first place; while the bells in third and sixth place change over. The cycle symbol is an operator which transforms the first change to the second, and such operators multiply together according to well-known rules. We will use capital letters to denote these transitions. Initially they will denote transitions between adjacent changes, but later on they will be used to denote transitions from one change to another which may occur elsewhere in the peal. Because of Rule ii, capital letters denoting transitions between adjacent changes are operators of period two. That is to say they are their own inverses. It is important to avoid confusing the change 1342 with the change generated by applying the operator (1342) to rounds, because this is 2413.

Singles. A peal on four bells is called *Singles*. If four bells hunt they produce the eight changes in the first block of Table I. A further move of a similar kind would restore rounds, so, in ringers' language, "the bell that the treble takes from lead makes second place and leads again; and the other bells dodge at the back-stroke lead of the treble." The mysteries of "hand-stroke" and "back-stroke" do not concern mathematicians; they arise because a bell swings two ways and successive strokes have to be rung with a different action. The meaning of "dodge" should be sufficiently clear from the table.

TABLE I

SINGLES

1 2 3 4	1 3 4 2	1 4 2 3
2 1 4 3	3 1 2 4	4 1 3 2
2 4 1 3	3 2 1 4	4 3 1 2
4 2 3 1	2 3 4 1	3 4 2 1
4 3 2 1	2 4 3 1	3 2 4 1
3 4 1 2	4 2 1 3	2 3 1 4
3 1 4 2	4 1 2 3	2 1 3 4
1 3 2 4	1 4 3 2	1 2 4 3

The process of hunting consists of employing the two operators $A = (12)(34)$ and $B = (23)$ alternately. These generate a group of order eight, the first eight

changes. In general hunting on n bells generates a group of order $2n$. $AB = (1342)$ and this element has period four. It is convenient to call these eight elements the *hunting sub-group* and to denote it by $\mathfrak{H}$. The last of these changes is 1324, and the peal continues by following this with the irregular move $C = (34)$, instead of employing B which would produce rounds. The second block therefore consists of the hunting sub-group premultiplied by $B^{-1}C = (243)$. This can be denoted by $(243)\mathfrak{H}$. After it has been rung the irregular move C has to be performed again, and the third block consists of $(243)^2\mathfrak{H}$. When the last change is followed once more by C we return to rounds because $(243)^3 = I$.

This method of ringing therefore displays the decomposition into cosets:

$$\mathfrak{G} = \mathfrak{H} + (243)\mathfrak{H} + (243)^2\mathfrak{H}.$$

Other methods are possible on four bells but their structure is not quite so clear as in this example.

Doubles. A peal on five bells is called *Doubles*. For five bells or more it is usually necessary to have a conductor who calls *bobs* and *singles* as required. That is to say he does the multiplications which change from one coset to the next—or to put the matter more correctly he does some of them, because the ringers engaged on any particular peal proceed under a set of standing orders which ensure that a number of the transitions from one coset to the next are done automatically. The first five bell method which we will consider is called *Grandsire Doubles*. The hunting sub-group of ten changes is generated by the two operations $A = (23)(45)$ and $B = (12)(34)$. In Grandsire Doubles the number of changes which the ringers can ring without further instructions is increased to 30 by using the further operation $C = (12)(45)$. In practice this irregular move is performed at the very beginning, before hunting commences; and the peal begins as shown in Table II.

TABLE II

GRANDSIRE DOUBLES

1 2 3 4 5		
2 1 3 5 4	2 1 5 4 3	2 1 4 3 5
2 3 1 4 5	2 5 1 3 4	2 4 1 5 3
3 2 4 1 5	5 2 3 1 4	4 2 5 1 3
3 4 2 5 1	5 3 2 4 1	4 5 2 3 1
4 3 5 2 1	3 5 4 2 1	5 4 3 2 1
4 5 3 1 2	3 4 5 1 2	5 3 4 1 2
5 4 1 3 2	4 3 1 5 2	3 5 1 4 2
5 1 4 2 3	4 1 3 2 5	3 1 5 2 4
1 5 2 4 3	1 4 2 3 5	1 3 2 5 4
1 2 5 3 4	1 2 4 5 3	1 2 3 4 5

These 30 changes are called *plain course*. It is easy to see why the ringers prefer to begin with the operation C. If this is done the return of the Treble to the lead can be taken as the cue to perform the irregular operation C again,

whereas if they commence with hunting no such clear indication is given them. It is now necessary to introduce some means whereby the ringers may pass from one plain course to another; and to do this the conductor calls bobs and singles. A *single* is a change which interrupts the regular work of an even number of bells, and a *bob* is a change which interrupts the regular work of an odd number. The distinction is again not one which a mathematician would have chosen. In some cases a bob or a single alters the parity of the changes which are being rung, whereas the other does not, but this is not the case in general.

Grandsire uses the single

$$\begin{bmatrix} 15432 \\ 15423 \end{bmatrix} = (45),$$

and Rule iii is relaxed at this point in this method of ringing. To ring a full peal of Grandsire, four bobs and two singles are necessary. Grandsire Doubles is a decomposition of the symmetric group on five symbols into twelve cosets of ten, masked by a pre-multiplication by C. Without the pre-multiplication plain course may be written:

$$\mathfrak{C} + (354)\mathfrak{C} + (354)^2\mathfrak{C}.$$

$(354) = B^{-1}C$ and is obviously of period three. This leads one to suspect that it is possible to generate a longer plain course by employing some move different from C. This can be done, and Plain Bob employs $D = (23)$ instead of C. $B^{-1}D = (1342)$, and so plain course consists of 40 changes, and the entire peal can be rung with fewer calls.

One of the most pleasing methods from a mathematical point of view is *Stedman's Doubles*. This method was invented round about 1640, and it displays a striking knowledge of decomposition into cosets very nearly one hundred years before Lagrange was born. Troyte [1] describes the method as follows: "Three bells go through the three bell changes while the other bells dodge behind; at the completion of each six changes one bell coming down from behind to take its part in the changes, and one going up behind to take its part in the dodging." In the Grandsire the work of the Treble is different from that of the other bells, as he pursues a hunting course the whole of the time, whereas the other bells modify their hunting course now and again. In Stedman's method the work of every bell is the same, and the symmetric group is decomposed into cosets of six. By tradition the method begins in the middle of a set, so the decomposition into cosets is pre-multiplied by three operations, or alternatively it is carried out from the reference order 32415 instead of 12345. The peal is shown in Table III. It will be noticed that the sets of six changes are rung as in our very first example, but taken forwards and backwards alternately.

The method requires two singles, and in this case they change the parity of the permutations being rung. The singles are indicated on the table by the letter S. They interchange some of the members of the cosets in which they occur, and they also decompose the whole group in another way—into two cosets of 60.

All the members of the alternating sub-group occur in the peal before and after the singles, while the members of the other coset occur between them.

TABLE III

STEDMAN'S DOUBLES

	1	2	3	4	5															
	2	1	3	5	4															
	2	3	1	4	5															
	3	2	4	1	5		5	2	3	4	1		3	1	4	2	5		5	1	3	4	2
	2	3	4	5	1		5	3	2	·	·		1	3	4	·	·		5	3	1	·	·
	2	4	3	1	5		3	5	2	·	·		1	4	3	·	·		3	5	1	·	·
	4	2	3	5	1		3	2	5	·	·		4	1	3	·	·		3	1	5	·	·
	4	3	2	1	5		2	3	5	·	·		4	3	1	·	·		1	3	5	·	·
	3	4	2	5	1		2	5	3	1	4		3	4	1	5	2		1	5	3	2	4
	4	3	5	2	1		5	2	1	3	4		4	3	5	1	2		5	1	2	3	4
S	4	5	3	1	2		2	5	1	·	·	S	4	5	3	2	1		1	5	2	·	·
	5	4	3	1	2		2	1	5	·	·		5	4	3	2	1		1	2	5	·	·
	5	3	4	2	1		1	2	5	·	·		5	3	4	1	2		2	1	5	·	·
	3	5	4	1	2		1	5	2	·	·		3	5	4	2	1		2	5	1	·	·
	3	4	5	2	1		5	1	2	4	3		3	4	5	1	2		5	2	1	4	3
	4	3	2	5	1		1	5	4	2	3		4	3	1	5	2		2	5	4	1	3
	3	4	2	1	5		1	4	5	·	·		3	4	1	·	·		2	4	5	·	·
	3	2	4	5	1		4	1	5	·	·		3	1	4	·	·		4	2	5	·	·
	2	3	4	1	5		4	5	1	·	·		1	3	4	·	·		4	5	2	·	·
	2	4	3	5	1		5	4	1	·	·		1	4	3	·	·		5	4	2	·	·
	4	2	3	1	5		5	1	4	3	2		4	1	3	2	5		5	2	4	3	1
	2	4	1	3	5		1	5	3	4	2		1	4	2	3	5		2	5	3	4	1
	2	1	4	5	3		5	1	3	·	·		1	2	4	·	·		5	2	3	·	·
	1	2	4	3	5		5	3	1	·	·		2	1	4	·	·		5	3	2	·	·
	1	4	2	5	3		3	5	1	·	·		2	4	1	·	·		3	5	2	·	·
	4	1	2	3	5		3	1	5	·	·		4	2	1	·	·		3	2	5	·	·
	4	2	1	5	3		1	3	5	2	4		4	1	2	5	3		2	3	5	1	4
	2	4	5	1	3		3	1	2	5	4		1	4	5	2	3		3	2	1	5	4
	4	2	5	3	1		3	2	1	·	·		4	1	5	·	·		3	1	2	·	·
	4	5	2	1	3		2	3	1	·	·		4	5	1	·	·		1	3	2	·	·
	5	4	2	3	1		2	1	3	·	·		5	4	1	·	·		1	2	3	4	5
	5	2	4	1	3		1	2	3	·	·		5	1	4	·	·						
	2	5	4	3	1		1	3	2	4	5		1	5	4	3	2						

The number of changes generated by some particular method can be calculated quite easily. Consider for example *Plain Bob Major*. (A peal on eight bells is called *Major*.) The complete 40,320 changes on eight bells were first rung at Leeds, Kent, in 1761, by relays of men who took 27 hours to complete the task. Plain Bob Major generates the hunting sub-group by means of $A=(12)(34)(56)(78)$ and $B=(23)(45)(67)$, and when the Treble returns to the lead, rounds are avoided by using $C=(34)(56)(78)$ which means the second coset commences with $B^{-1}C=(2468753)$ which is of period 7. Plain course therefore consists of 7

cosets of 16, which is 116 changes. This method uses the bob (23)(56)(78) and the single (56)(78), and it is interesting to work out how many more changes these will generate and where they must be used.

Treble Bob. The *Treble Bob* methods of ringing are more complicated than the plain methods because the bells do not follow a simple hunting course, but dodge on their way up and down. This means that a "lead" (the set of changes between successive appearances of the Treble as the first bell) is twice as long as in the plain method, and that the changes of a lead do not possess the group property, being usually a complex generated by three operations employed according to some scheme such as $ABACABAC\cdots$. Even here group properties are very apparent in the tables which are given in books on campanology. The move at the end of a lead in Treble Bob is arranged so that when the Treble returns to the front the other $n-1$ bells are cyclically permuted; in this way plain course consists of as many leads as possible.

We have so far pointed out only that the three blocks of Table I are a decomposition into left-hand cosets. If we read the table by lines we see a decomposition into right-hand cosets using a sub-group of order three. This can also be seen in Table II although it is a little obscured by the irregular way in which the peal begins, and it is a general feature of Plain and Treble Bob methods.

As we have indicated earlier most of the methods of campanology are empirical, *ad hoc* methods devised some hundreds of years ago long before the invention of formal group theory, but the subject has not been without its great theorists. One of the finest of these was W. H. Thompson, who was not a practicing ringer but a civil servant in India who made a hobby of solving change ringing problems. As far as can be ascertained he was not a mathematician and his writings [3] do not suggest that he realized that the calculations which he was performing were in fact pieces of group theory. He was concerned in particular with the following problem: "Is it possible to ring all the changes on seven bells (Triples) using the Grandsire method with only plain and bob leads?" He devised his own notation for this problem, and it bears striking analogies to the notation invented by the celebrated Irish mathematician Hamilton for another problem [4] a few years earlier.

Because the work in each lead is perfectly regular (being the hunting subgroup), the analysis of a complicated peal is greatly simplified by considering only the "lead-ends." Putting this in a way which seems more natural to mathematicians we need only consider the first member of each coset. The effect of a plain lead in Grandsire Triples is to send bells which have just rung a lead starting with rounds into the next lead in the order 1253746. We may therefore write $P=(34675)$. The effect of a bob lead is to send bells which have just rung a lead starting with rounds into the next lead in the order 1752634, and so we put $B=(247)(365)$.

There are 7! changes in Grandsire Triples and each lead contains 14. The total number of leads is therefore 6!/2 and the initial members of the cosets form the alternating group on six symbols. Denoting these six symbols by 2, 3, 4,

5, 6, 7 (because bell number 1 is always in front at the beginning and end of a lead) the question which Thompson was discussing was: "Will P and B generate the alternating group on six symbols, and furthermore generate it unicursally?"

The answer was "No," but the interest is in Thompson's methods. Using the notion of lead-ends he drew up a chart in such a way that the unicursal generation of the group became a matter of the unicursal description of a polytope whose edges represented the leads of the peal. The problem is remarkably like the problem of the game of visiting the vertices of an icosahedron which Hamilton had discussed a few years before [4]. Hamilton's problem is by far the simpler of the two, and it is not exactly the same as he is concerned with visiting the vertices of the polytope and Thompson is concerned with traversing the edges, but the two employ an algebraic notation which is almost identical. The solution of Hamilton's game presumably gives a peal on six bells (or rather a *touch*, because the 20 vertices of the icosahedron correspond to only 40 leads, whereas 60 leads are necessary to ring a full peal in Minor).

After constructing the polytope Thompson argues as follows. If the result of following x with a plain ending is the same as the result of following y with a bob, then clearly the leads x and y must be treated in the same way, that is both must be plain or both must be bob. Now if $xP=yB$ then $y=xPB^{-1}$. Therefore if x is plain or bob, $x(PB^{-1})$ must be the same. But the operator PB^{-1} is of period five, and so all the leads may be split up into sets of five which must all be treated the same way. Thompson called these Q-sets, and they are of course cosets in the strict meaning of the term.

If all the 360 leads are plain they are split up into 72 *round blocks* since P is of period five. On the other hand if they are all bobs they are split up into 120 round blocks since B is of period three. The peal can be rung if we may start at one of these extremes and by bobbing or plaining suitable Q-sets convert the whole 360 leads into one round block. This would be denoted by a chain of letters of the form *PPBPBPPPB etc*, where the whole product was equal to the identity but no sub-section was. Thompson showed this to be impossible by proving that bobbing or plaining any Q-set always results in the loss or gain of an even number of round blocks. Hence the number of round blocks, which is 72 or 120 to start with, can never be reduced to one. The beauty of the proof is marred by the fact that the stage showing that the number of round blocks lost or gained is always even is carried out by a long and tedious process of enumeration. But it is very difficult to see any means by which this could have been avoided. The enumeration of the cosets of a group of large order is inevitably tedious, and modern processes do not seem to offer any way of reducing Thompson's labours to any marked extent.

The study of campanology therefore shows how bell-ringers acquired a considerable amount of empirical knowledge about groups long before the theoretical basis was established by mathematicians, and it also provides instructive examples of the abstract theorems. The main problem of campanology is a problem of Group Theory to which no practical solution seems to have been

found even yet: "Given a particular set of elements of a group, will they generate the whole group, and will they generate it unicursally?"

Bibliography

1. Troyte, C. A. W. Change Ringing, London, 1869.
2. Morris, E. History and Art of Change Ringing, 1931.
3. Thompson, W. H. A Note on Grandsire Triples, London, 1886. (This is reprinted in Grandsire, J. W. Snowden, London, 1905.)
4. Hamilton, W. R. Quarterly Journal of Mathematics, London, 1862, vol. V, p. 305; or Philosophical Magazine, January, 1884, series 5, vol. XVII, p. 42. (This is also described in W. W. Rouse Ball, Mathematical Recreations and Essays, Eleventh Edition, London, 1939.)

POLYNOMIALS DEFINED BY GENERATING RELATIONS

R. P. BOAS, JR., AND R. C. BUCK,* Northwestern University and University of Wisconsin

1. Introduction. Many familiar sets of polynomials $p_n(z)$ have generating relations of the form

$$\sum_{n=0}^{\infty} w^n p_n(z) = A(w)\psi(zg(w)), \tag{1}$$

where $A(w)$, $\psi(t)$, $g(w)$ are power series. These sets include such general classes as Appell polynomials ($g(w)=w$, $\psi(t)=e^t$), Sheffer polynomials† ($\psi(t)=e^t$), Brenke polynomials [1] ($g(w)=w$), and such particular sets as certain constant multiples of the Jacobi,‡ Laguerre and Hermite polynomials. The generating relation (1) appears not to have been discussed before in full generality. Here we shall characterize the polynomials defined by (1) by means of both an explicit formula and a recursion relation; the explicit formula reduces to known results for Appell and Brenke polynomials, but otherwise seems to be new. We also give a brief discussion of a simple special subclass that is interestingly related to the class of Appell polynomials.

2. An explicit formula. Let $A(w)$, $\psi(t)$, $g(w)$ be the formal power series

$$A(w) = \sum_{n=0}^{\infty} a_n w^n, \qquad a_0 \neq 0; \tag{2}$$

* This investigation was supported in part by the Office of Ordnance Research.

† Sheffer [5], [6] calls them sets of type zero; Steffensen [9], [10] calls them poweroids; the Bateman Manuscript Project [2] calls them sets of generalized Appell type.

‡ The Jacobi polynomials $P_n^{(\alpha,\beta)}(z)$ have a generating relation of the form (1) only when $\alpha-\beta=-1$, 0, or 1 (Smith [8]), but $P_n^{(\alpha,\beta)}(x+1)$ do have a generating relation (1) (Erdélyi [3], p. 264).

$$\psi(t) = \sum_{n=0}^{\infty} \gamma_n t^n, \tag{3}$$

with $\psi(t)$ not a polynomial;

$$g(w) = \sum_{n=0}^{\infty} b_n w^n. \tag{4}$$

We are actually concerned only with the algebraic relations among coefficients implied by relations among formal power series. In establishing them we may operate with the power series as if they were convergent; for a discussion of this point see Sheffer [5]. In all the applications of which we are aware, the power series have positive radii of convergence.

THEOREM 1. *In* (1), *the* $p_n(z)$ *are polynomials if and only if* $b_0 = 0$; *if this is the case,* $p_n(z)$ *is of degree* n *if and only if* $b_1\gamma_n \neq 0$; *moreover,*

$$p_n(z) = \sum_{j=0}^{n} z^j \gamma_j \sum a_{k_0} b_{k_1} b_{k_2} \cdots b_{k_j}, \tag{5}$$

where the inner summation extends over all sets of $j+1$ *non-negative integers* $\{k\}$ *such that* $k_0 + k_1 + \cdots + k_j = n$.

In particular, if $g(w) = w$ this reduces to Brenke's formula [1]

$$p_n(z) = \sum_{j=0}^{n} z^j \gamma_j a_{n-j},$$

which in turn reduces to the well-known formula for Appell polynomials,

$$p_n(z) = \sum_{j=0}^{n} z^j a_{n-j}/j!, \tag{6}$$

if in addition $\psi(t) = e^t$. In the general case with $\psi(t) = e^t$ we have the following formula for Sheffer polynomials:

$$p_n(z) = \sum_{j=0}^{n} \frac{z^j}{j!} \sum a_{k_0} b_{k_1} \cdots b_{k_j}, \qquad k_0 + k_1 + \cdots + k_j = n.$$

This could also be deduced from Sheffer's integral expression for these polynomials [7].

Let (1) hold with A, ψ, g from (2), (3), (4), and write $K(z, w) = A(w)\psi(zg(w))$. Let $p_n(z) = \sum_{j=0}^{\infty} p_{nj} z^j$. Then

$$K(z, w) = \sum_{n=0}^{\infty} w^n \sum_{j=0}^{\infty} p_{nj} z^j = \sum_{j=0}^{\infty} z^j \sum_{n=0}^{\infty} p_{nj} w^n,$$

so that

$$k! \sum_{n=0}^{\infty} p_{n,j} w^n = \left. \frac{\partial^j K}{\partial z^j} \right|_{z=0}.$$

We also have

$$\frac{\partial^j K}{\partial z^j} = A(w)\{g(w)\}^j \psi^{(j)}(zg(w)),$$

and so

$$\sum_{n=0}^{\infty} p_{n,j} w^n = \gamma_j A(w)\{g(w)\}^j. \tag{7}$$

Now if $g(0)=0$, the right-hand side has a power series beginning with $\gamma_j a_0 b_1^j w^j$, so that $p_{n,j}=0$ at least for $n \leqq j$ and so $p_n(z)$ is a polynomial of degree n at most. On the other hand, if $g(0)=b_0 \neq 0$, it follows from (7) that, for an infinity of integers j, $p_{0,j}=\gamma_j a_0 b_0^j$ does not vanish, so that $p_0(z)$ is not a polynomial.

Now suppose that $g(0)=0$ and write (7) in the form

$$\sum_{n=j}^{\infty} p_{n,j} w^n = \gamma_j A(w)\{g(w)\}^j. \tag{8}$$

Picking out the coefficient of w^k on both sides, we have $p_{j,j}=\gamma_j a_0 b_1^j$, which shows that $p_j(z)$ is of degree j if and only if $\gamma_j b_1 \neq 0$. Similarly, $p_{n,j}$ is the coefficient of w^n in the product of the $j+1$ power series on the right-hand side of (8), and so

$$p_{n,j} = \gamma_j \sum a_{k_0} b_{k_1} \cdots b_{k_j}, \qquad k_0 + k_1 + \cdots + k_j = n.$$

This completes the proof of Theorem 1.

With any polynomial $p_n(z)=c_0 z^n + c_1 z^{n-1} + \cdots + c_n$ we can associate another polynomial $p_n^*(z)=c_0 + c_1 z + \cdots + c_n z^n = z^n p_n(1/z)$; we call this the reversed polynomial, for lack of a generally accepted term. For the special case of Brenke polynomials we have the following result.

THEOREM 2. *If $p_n(z)$ are Brenke polynomials with $\gamma_0 \neq 0$ and $A(w)$ not a polynomial, $p_n^*(z)$ are also Brenke polynomials.*

In fact, we have

$$A(w)\psi(zw) = \sum p_n(z) w^n,$$

and writing $1/z$ for z and then zw for w we obtain

$$A(zw)\psi(w) = \sum p_n^*(z) w^n.$$

3. Characterization by properties of the generating function. We next obtain a characterization of polynomial sets which are generated by relations of the form (1). Given any sequence $\{p_n(z)\}$ with p_n of degree n, we form the series

$$\sum_{n=0}^{\infty} w^n p_n(z) = K(z, w).$$

Our first characterization is in terms of properties of K.

THEOREM 3. *A necessary and sufficient condition that* $K(z, w) = A(w)\psi(zg(w))$ *with* A, ψ, g *as in* (2), (3), (4) *and* $b_0 = 0$, $b_1 = 1$, *is that there exist power series*

$$\alpha(w) = \sum_{n=0}^{\infty} \alpha_n w^n,$$

$$\beta(w) = 1 + \sum_{n=1}^{\infty} \beta_n w^n,$$

(9)

such that

$$K_2(z, w) = \alpha(w)K(z, w) + zw^{-1}\beta(w)K_1(z, w), \tag{10}$$

where the subscripts denote partial derivatives.

If $K(z, w)$ has the specified form we can verify by differentiation that (10) holds with $\alpha(w) = A'(w)/A(w)$, $\beta(w) = wg'(w)/g(w)$.

Conversely, if (10) holds, choose $A(w)$ of the form (2) so that $A'(w)/A(w) = \alpha(w)$. Set $H(z, w) = K(z, w)/A(w)$. Since $K_1(z, w) = A(w)H_1(z, w)$ and $K_2(z, w) = A'(w)H(z, w) + A(w)H_2(z, w)$, (10) implies that

$$H_2(z, w) = zw^{-1}\beta(w)H_1(z, w). \tag{11}$$

Now choose $g(w)$ of the form (4) so that $\beta(w) = wg'(w)/g(w)$, and so that $g(0) = 0$, $g'(0) = 1$. Then (11) states that

$$H_2(z, w) = \{zg'(w)/g(w)\}H_1(z, w).$$

Replacing z by $z/g(w)$, we obtain

$$H_2\{z/g(w), w\} - zg'(w)\{g(w)\}^{-2}H_1\{z/g(w), w\} = 0,$$

which is to say

$$\frac{\partial}{\partial w} H\{z/g(w), w\} = 0.$$

Thus $H\{z/g(w), w\}$ is independent of w, and so has the form $\psi(z)$ for some ψ. In other words,

$$K(z, w) = A(w)H(z, w) = A(w)\psi(zg(w)).$$

4. Characterization by a recursion relation. Theorem 3 can be expressed more directly in terms of the $p_n(z)$ as follows.

THEOREM 4. *If* $p_n(z)$ *is a polynomial of degree* n, *the sequence* $\{p_n(z)\}$ *has a generating relation of the form* (1) *with* A, ψ, g *as in* (2), (3), (4) *and* $b_0 = 0$, $b_1 = 1$, *if and only if there are two sequences of numbers* $\{\alpha_n\}$ *and* $\{\beta_n\}$ *such that*

$$z^{n+1}\frac{d}{dz}\{z^{-n}p_n(z)\} + U_0[p_{n-1}(z)] + U_1[p_{n-2}(z)] + \cdots + U_{n-1}[p_0(z)] = 0, \tag{12}$$

where U_k is the linear differential operator

$$U_k = \alpha_k + z\beta_{k+1}\frac{d}{dz}.$$

More explicitly, (12) states that

$$\begin{aligned}
z^2[p_1(z)/z]' &= -\alpha_0 p_0(z),\\
z^3[p_2(z)/z^2]' &= -[\alpha_1 p_0(z) + \alpha_0 p_1(z)] + \beta_1 z p_1'(z),\\
z^4[p_3(z)/z^3]' &= -[\alpha_2 p_0(z) + \alpha_1 p_1(z) + \alpha_0 p_2(z)] + z[\beta_2 p_1'(z) + \beta_1 p_2'(z)],\\
&\cdots\cdots\cdots

so that, for $z=0$, $KK_{11}=c\{K_1\}^2$, where c is a constant. When $K(z, w)=H\{a(w)+zb(w)\}$, this leads to the equation

$$H(a(w))H''(a(w))\{b(w)\}^2 = c\{H'(a(w)\}^2\{b(w)\}^2.$$

The only solutions of this with $H(0)=1$ are $H(t)=e^{\alpha t}$ and $H(t)=(1+\alpha t)^{-\lambda}$. The number α can be absorbed into the functions $a(w)$ and $b(w)$, so that we may take $\alpha=1$ or $\alpha=-1$, respectively.

Thus, in addition to the Sheffer polynomial sets, which correspond to the first choice $H(t)=e^t$, one is naturally led to consider a second class of sets whose generating functions are $\{1-a(w)-zb(w)\}^{-\lambda}$. Putting this into the standard form (1) we have

$$A(w)\{1 - zg(w)\}^{-\lambda} = \sum_{n=0}^{\infty} p_n(z)w^n.$$

The special case $g(w)=w$ yields polynomial sets which are in many ways analogous to the Appell polynomials. They are defined by

$$A(w)(1 - zw)^{-\lambda} = \sum_{n=0}^{\infty} P_n(z)w^n \tag{16}$$

and satisfy the mixed equation

$$P'_{n+1}(z) = \lambda P_n(z) + zP'_n(z).$$

To see the connection between these polynomials and those of Appell, set $Q_n(z)=P_n(z/\lambda)$. Then

$$Q'_{n+1}(z) = Q_n(z) + (z/\lambda)Q'_n(z),$$

so that $\lim_{\lambda\to\infty} Q_n$ is the Appell polynomial of degree n associated with the function $A(w)$.

Some other interconnections may be mentioned. For example, formula (5) gives the explicit formula

$$P_n(z) = \sum_{j=0}^{n} a_{n-j}\binom{j+\lambda-1}{\lambda-1}z^j,$$

where $A(w)=\sum a_k w^k$. The reversed set, $P_n^*(z)=z^nP_n(1/z)$, is recognized to be exactly the nth Cesàro partial sum of order $\lambda-1$ (if $\lambda>0$) of the power series for $A(w)$ (see [3], p. 96). In particular, when $\lambda=1$, $P_n^*(z)=\sum_{k=0}^{n} a_k z^k$, so that $P_n(z)=\sum_{j=0}^{n} a_{n-j}z^j$ (compare (6)). In this case, there is a further formal connection with the Appell polynomials. The polynomials P_n obey the relation

$$\frac{P_{n+1}(z) - P_{n+1}(0)}{z} = P_n(z),$$

which is the special case $r=0$ of the formula

$$\frac{P_{n+1}(z) - P_{n+1}(rz)}{z - rz} = P_n(z). \tag{17}$$

When $r=1$, (17) becomes $P'_{n+1}(z)=P_n(z)$, the recursion relation for Appell polynomials. Thus the polynomial sets characterized by (17) form a family which contains both the Appell sets and the sets $\{P_n\}$ of (16) for $\lambda=1$. They have been investigated in detail by Sharma and Chak [4]; the recursion (17) was suggested to us by T. S. Motzkin several years ago. These polynomials are also a subclass of the general polynomials defined by (1); indeed, they are the Brenke polynomials with

$$\psi(w) = \sum_{n=0}^{\infty} \frac{w^n(r-1)^n}{(r^n-1)(r^{n-1}-1)\cdots(r-1)}.$$

References

1. W. C. Brenke, On generating functions of polynomial systems, this MONTHLY, vol. 52, 1945, pp. 297–301.

2. A. Erdélyi, editor, Higher Transcendental Functions, vol. 3, New York, 1955.

3. G. H. Hardy, Divergent Series, Oxford, 1949.

4. A. Sharma and A. M. Chak, The basic analogue of a class of polynomials, Riv. Mat. Univ. Parma, vol. 5, 1954, pp. 325–337.

5. I. M. Sheffer, Some properties of polynomial sets of type zero, Duke Math. J., vol., 5, 1939, pp. 590–622.

6. I. M. Sheffer, Some applications of certain polynomial classes, Bull. Amer. Math. Soc., vol. 47, 1941, pp. 885–898.

7. I. M. Sheffer, Note on Appell polynomials, Bull. Amer. Math. Soc., vol. 51, 1945, pp. 739–744.

8. R. C. T. Smith, Generating functions of Appell form for the classical orthogonal polynomials, Proc. Amer. Math. Soc., vol. 7, 1956, pp. 636–641.

9. J. F. Steffensen, The poweroid, an extension of the mathematical notion of power, Acta Math., vol. 73, 1941, pp. 333–366.

10. J. F. Steffensen, On the polynomials $R_\nu^{[\lambda]}(x)$, $N_\nu^{[\lambda]}(x)$ and $M_\nu^{[\lambda]}(x)$, Acta Math., vol. 78, 1946, pp. 291–314.

A NOTE ON QUASI-IDEMPOTENT MATRICES*

C. R. MARATHE, Muslim University, Aligarh, India

1. Introduction. Let A be a square matrix of order p over the field C of complex numbers, and $F(x)$ a polynomial matrix of the same order p over the

* I am thankful to the referee for his comments and suggestions.

My thanks are also due to Professor S. M. Shah for his guidance in the preparation of this note.

ring $C[x]$ of polynomials in the indeterminate x. Huff [1] defines A as quasi-idempotent if $A^r = F(r)$ for all positive integral r.

He states that by substituting fractional and negative values for r in $F(r)$ we can obtain the corresponding fractional and negative powers of A. It is, therefore, of some interest to know what matrices can possibly be quasi-idempotent.

Two necessary and sufficient conditions to determine a quasi-idempotent matrix are given by Huff. A few more results about such matrices are obtained in this note.

2. Definitions. Huff has given some definitions which are stated below for the sake of convenience.

(i). If A is q.i.p. (read quasi-idempotent) such that $A^r = F(r)$, then $F(x)$ is called "an exponential polynomial matrix" associated with A.

(ii). Clearly $F(x) = \sum A_k x^k$ where A_k are square matrices over C of the same order as that of A. Thus $F(x)$ is a polynomial in the indeterminate x with *matric* coefficients. If we replace x by a square matrix X we get a *matric* polynomial with *matric* coefficients. Huff calls the latter the *image* of the first. There is a natural isomorphism between $(C[X])_p$ and $C_p[X]$. Thus if $F(x) = A_0 + A_1 x + A_2 x^2 + \cdots + A_k x^k$, then the image $F^*(X)$ of $F(x)$ is given by

$$F^*(X) = A_0 + A_1 X + A_2 X^2 + \cdots + A_k X^k,$$

and $F^*(X)$ is called "the exponential matric polynomial" associated with A.

(iii). If A is q.i.p. a necessary and sufficient condition proved by Huff is $A(A-I)^k = 0$ for some positive integer k, I being the unit matrix of the same order as A. Further, if n is such that $A(A-I)^n \neq 0$ but $A(A-I)^{n+1} = 0$, then n is called the *index* of the q.i.p. matrix A.

(iv). If A is q.i.p., Huff has shown that there is an "identity" matrix B_0 for A such that $AB_0 = B_0A = A$ and $(A - B_0)^{n+1} = 0$. In case A is non-singular, B_0 equals I.

3. Uniqueness and existence.

THEOREM 1. *Given a q.i.p. matrix A, there is a unique exponential polynomial matrix $F(x)$ associated with A, and conversely, given an exponential polynomial matrix $F(x)$, there is a unique q.i.p. matrix A associated with $F(x)$.*

Proof. For a given q.i.p. matrix A the existence of a unique $F^*(X)$ is stated by Huff. Now there is a natural isomorphism between $(C[X])_p$ and $C_p[X]$. Hence uniqueness of $F^*(X)$ implies the uniqueness of $F(x)$.

The converse is obvious. For, if possible, let there be two q.i.p. matrices A and B associated with the same exponential polynomial matrix $F(x)$. Hence $A^r = F(r) = B^r$ for all positive integers r. In particular, when r equals 1 we get $A = B$.

4. The minimum equation of a Q.I.P. matrix.

THEOREM 2. *The minimum equation of every q.i.p. matrix has only two possible distinct roots viz.* 0 *and* 1.

Proof. Let A be q.i.p. Then for some positive integer k we have:

$$A(A - I)^k = 0.$$

Thus A satisfies $\lambda(\lambda-1)^k=0$. Hence if $m(\lambda)=0$ is the minimum equation of A then $m(\lambda)\,|\,\lambda(\lambda-1)^k$. Hence $m(\lambda)$ can have only two distinct factors λ and $(\lambda-1)$. Hence the possible distinct roots of $m(\lambda)=0$ are zero and unity.

THEOREM 3. *The zero root of the minimum equation of a q.i.p. matrix of index n is at most simple and the multiplicity of the unit root is* $(n+1)$.

Proof. If A is q.i.p. and n its index then by the definition of the index $A(A-I)^n\neq 0$ and $A(A-I)^{n+1}=0$. So that $m(\lambda)\nmid\lambda(\lambda-1)^n$ but $m(\lambda)\,|\,\lambda(\lambda-1)^{n+1}$. Hence $m(\lambda)$ can at best have one nonrepeated factor λ and the factor $(\lambda-1)$ is repeated $(n+1)$ times. This proves the theorem.

It has been observed by Huff that if a q.i.p. matrix A is nonsingular then its identity matrix B_0 is the universal identity matrix I. This also follows from the above theorem. For if A is nonsingular and q.i.p. then the minimum equation $m(\lambda)=0$ of A has no zero root. Hence $(\lambda-1)^{n+1}=0$ corresponds to $(A-I)^{n+1}=0$. But also $(A-B_0)^{n+1}=0$. So $B_0=I$.

5. Transforms of Q.I.P. matrices.

THEOREM 4. *Let A be q.i.p. of index n and order p and let $B=HAH^{-1}$ where H is a $p\times p$ nonsingular matrix over C. Then B is q.i.p. of index n.*

Proof. If $f(x)$ is any polynomial with coefficients in C then $f(B)=f(HAH^{-1})=Hf(A)H^{-1}$. Put $f(x)=x(x-1)^{n+1}$. Since A is q.i.p. so $f(A)=0$. Hence $f(B)=0$. Then B is q.i.p. Again as $A(A-I)^n\neq 0$ we have $B(B-I)^n=HA(A-I)^nH^{-1}\neq 0$ since H is not a divisor of zero. Hence the index of B is n.

In the above theorem H is nonsingular. This theorem can be slightly generalized even if H is singular.

Let H be singular but q.i.p. with $F(x)$ as its associated exponential polynomial matrix and G_0 as its identity. Further let G_0 be commutative with A. Now H^{-1} can be defined as $H^{-1}=F(-1)$ and $HH^{-1}=F(0)=G_0$. Hence $f(B)=f(HAH^{-1})=Hf(A)H^{-1}$. So that if A is q.i.p. so is B. However, the index of B may not be the same as that of A since H may be a divisor of zero.

THEOREM 5. *If B is the transform of a q.i.p. matrix A, the matrix of transformation being H, then the exponential matric polynomial $G^*(X)$ of B is also the transform by H of the exponential matric polynomial $F^*(X)$ of A.*

Proof. Let $F^*(X)=F_0+F_1X+\cdots+F_kX^k$ and let $G^*(X)=G_0+G_1X+\cdots+G_kX^k$. Let $B=HAH^{-1}$. Hence $H^{-1}BH=A$. Now $A^r=F(r)$ for all

positive integers r. So $B^r = HF^*(rI)H^{-1}$. But the exponential polynomial matrix is unique by Theorem 1. Hence $G(r) = HF^*(rI)H^{-1}$. Thus the degree of $G(x)$ is the same as the degree of $F(x)$ *i.e.*, $h = k$. Further $G_i = HF_iH^{-1}$ for $i = 0, 1, 2, \cdots, h = k$. Hence we have

$$G(x) = H(F_0 + F_1x + \cdots + F_kx^k)H^{-1}$$

so that

$$\begin{aligned} G^*(X) &= H(F_0 + F_1X + \cdots + F_kX^k)H^{-1} \\ &= HF^*(X)H^{-1} \end{aligned}$$

which proves the theorem.

6. **Characteristic roots of Q.I.P. matrices.** The minimum equation of a q.i.p. matrix is not in general its characteristic equation. For example consider the unit matrix I of order p. As I is idempotent it is also q.i.p. The minimum equation of I is $\lambda - 1 = 0$, while its characteristic equation is $(\lambda - 1)^p = 0$.

THEOREM 6. *The possible distinct characteristic roots of a q.i.p. matrix A are zero and unity. The multiplicity of the unit root equals the rank of A while the multiplicity of the zero root equals its nullity.*

Proof. The distinct characteristic roots of a matrix are the distinct roots of its minimum equation. The minimum equation of a q.i.p. matrix has only two distinct roots *viz.*, zero and/or unity. So the distinct characteristic roots of a q.i.p. matrix are possibly zero and unity only.

If A is q.i.p. of order n and nonsingular then there is no zero root of the characteristic equation. Hence the multiplicity of the unit root is n, the rank of A.

If A is singular of rank $\rho = n - \sigma$, where σ is its nullity, then the characteristic roots of A contain σ zero roots and the remaining roots are non-zero. Now as A is q.i.p. all the remaining roots are unity, their number being ρ, which proves the theorem.

The converse of the above theorem is not necessarily true. A matrix may have distinct characteristic roots zero and/or unity and yet it may not be q.i.p.

For example

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

has distinct characteristic roots zero and unity. Now the minimum equation of A is $\lambda^2(\lambda - 1) = 0$, and so by Theorem 3 is not q.i.p.

Reference

[1]. G. B. Huff, On quasi-idempotent matrices, this MONTHLY, vol. 62, 1955, pp. 334–339.

ALGEBRAIC PROPERTIES OF CLASSES OF FUNCTIONS

A. A. GRAU AND B. T. GOLDBECK, JR., University of Oklahoma

The concepts of odd and even functions are well known. An even function is a function $f(x)$ such that $f(-x)=f(x)$, and an odd function is a function $f(x)$ such that $f(-x)=-f(x)$. It is also well known that any function may be uniquely expressed as the sum of an odd and even function [1].

A simple generalization of these concepts to n types of functions is furnished by the nth roots of unity. Let ω be a primitive nth root of unity. The distinct nth roots of unity are ω^j, $j=0, 1, \cdots, (n-1)$ [2].

Consider a function, $f(x)$, of a complex variable, x, to be of type (n, k) with $0 \leqq k < n$ if

$$f(\omega x) = \omega^k f(x).$$

It follows, that for each n, $f(x)$ may belong, at most, to one of these types; for let

$$f(\omega x) = \omega^k f(x),$$

and

$$f(\omega x) = \omega^s f(x).$$

Then

$$(\omega^k - \omega^s) f(x) = 0,$$

or

$$\omega^k = \omega^s.$$

First, a lemma is proved.

LEMMA. *If ω is a primitive nth root of unity, then the determinant $D \neq 0$, where*

$$D = \begin{vmatrix} 1 & 1 & 1 & \cdots & 1 & 1 \\ 1 & \omega & \omega^2 & & \omega^{(n-2)} & \omega^{(n-1)} \\ 1 & \omega^2 & \omega^4 & & \omega^{2(n-2)} & \omega^{2(n-1)} \\ \cdot & \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & \cdot & & \cdot & \cdot \\ 1 & \omega^{n-2} & \omega^{2(n-2)} & \cdots & \omega^{(n-2)^2} & \omega^{(n-2)(n-1)} \\ 1 & \omega^{n-1} & \omega^{2(n-1)} & \cdots & \omega^{(n-1)(n-2)} & \omega^{(n-1)^2} \end{vmatrix}.$$

Proof. It may be verified that

$$D^2 = \begin{vmatrix} n & 0 & 0 \cdots 0 & 0 \\ 0 & 0 & 0 \cdots 0 & n \\ 0 & 0 & 0 \cdots n & 0 \\ \cdot & \cdot & \cdot \quad \cdot & \cdot \\ \cdot & \cdot & \cdot \quad \cdot & \cdot \\ \cdot & \cdot & \cdot \quad \cdot & \cdot \\ 0 & 0 & n \cdots 0 & 0 \\ 0 & n & 0 \cdots 0 & 0 \end{vmatrix}$$

$$= (-1)^{(n-2)(n-1)/2} n^n \neq 0.$$

This result determines the value of D to within sign. This cannot be predetermined, since this depends upon the primitive root of unity used.

THEOREM 1. *If $f(x)$ is any complex valued function, it may be expressed uniquely as*

$$f(x) = f_0(x) + f_1(x) + f_2(x) + \cdots + f_{n-1}(x),$$

where $f_k(x)$ is of type (n, k).

Proof. If $f(x)$ has an expansion of this type, then

$$f(\omega^j x) = f_0(x) + \omega^j f_1(x) + \omega^{2j} f_2(x) + \cdots + \omega^{(n-1)j} f_{n-1}(x),$$

for $j=0, 1, 2, \cdots, (n-1)$.

Since the determinant of the coefficients of $f_0(x), f_1(x), \cdots, f_{n-1}(x)$ is D and therefore non-zero by the lemma, these functions have a unique expansion in terms of $f(x), f(\omega x), \cdots, f(\omega^{(n-1)}x)$ and are, therefore, unique.

It may be easily verified that these expansions are:

$$f_j(x) = \frac{1}{n}\left[f(x) + \omega^j f(\omega x) + \omega^{2j} f(\omega^2 x) + \cdots + \omega^{(n-1)j} f(\omega^{n-1} x)\right]$$

for $j=0, 1, 2, \cdots, (n-1)$.

The following isomorphism can be established between a system of non-empty classes, ${}_n\mathfrak{F}_k$, of the functions of type (n, k), and the ring of residue classes of integers modulo n.

With suitable definitions of operations, the classes ${}_n\mathfrak{F}_k$ form a ring. Let ${}_n\mathfrak{F}_k$ be the set of all functions of type (n, k), and let ${}_n\mathfrak{F}_k \cdot {}_n\mathfrak{F}_s$ denote the set of all functions which arise as products of functions in ${}_n\mathfrak{F}_k$ and ${}_n\mathfrak{F}_s$. Also, if $f(x) \in {}_n\mathfrak{F}_k$, $g(x) \in {}_n\mathfrak{F}_s$, and $f[g(x)] \in {}_n\mathfrak{F}_{ks}$, it is possible, then, to define the *composition* of the classes ${}_n\mathfrak{F}_k$ and ${}_n\mathfrak{F}_s$, ${}_n\mathfrak{F}_k \circ {}_n\mathfrak{F}_s$ to be the class ${}_n\mathfrak{F}_{ks}$.

Let $f(x) \in {}_n\mathfrak{F}_k$ and $g(x) \in {}_n\mathfrak{F}_s$; then

$$f(\omega x) \cdot g(\omega x) = \omega^k f(x) \cdot \omega^s g(x)$$
$$= \omega^{k+s} f(x) g(x).$$

Hence,

$$ {}_n\mathfrak{F}_k \cdot {}_n\mathfrak{F}_s \subset {}_n\mathfrak{F}_{k+s}. $$

Conversely, if $f(x) \in {}_n\mathfrak{F}_{k+s}$, then

$$ f(x) = x^k[x^{-k}f(x)], $$

and

$$ x^k \in {}_n\mathfrak{F}_k $$
$$ [x^{-k}f(x)] \in {}_n\mathfrak{F}_s. $$

Therefore, ${}_n\mathfrak{F}_k \cdot {}_n\mathfrak{F}_s \supset {}_n\mathfrak{F}_{k+s}$ and

$$ {}_n\mathfrak{F}_k \cdot {}_n\mathfrak{F}_s = {}_n\mathfrak{F}_m, \qquad (k + s \equiv m \bmod n). $$

Then, if $f(x) \in {}_n\mathfrak{F}_k$ and $g(x) \in {}_n\mathfrak{F}_s$,

$$ f[g(\omega x)] = f[\omega^s g(x)] $$
$$ = \omega^{ks} f[g(x)]. $$

From this and the definition, it follows:

$$ {}_n\mathfrak{F}_k \circ {}_n\mathfrak{F}_s = {}_n\mathfrak{F}_m, \qquad (ks \equiv m \bmod n). $$

THEOREM 2. *The ring of classes of functions, ${}_n\mathfrak{F}_k$, is isomorphic to the ring of residue classes of integers modulo n.*

Proof. Define a mapping h, such that

$$ h: {}_n\mathfrak{F}_k \leftrightarrow k \bmod n. $$

Then,

$$ h: {}_n\mathfrak{F}_k \cdot {}_n\mathfrak{F}_s = {}_n\mathfrak{F}_{k+s} \leftrightarrow (k + s) \bmod n; $$
$$ h: {}_n\mathfrak{F}_k \circ {}_n\mathfrak{F}_s = {}_n\mathfrak{F}_{ks} \leftrightarrow (ks) \bmod n. $$

Therefore, h is an isomorphism between the ring of classes of functions, ${}_n\mathfrak{F}_k$, and the ring of residue classes of integers modulo n.

In view of the fact that the residue classes modulo 2 are the odd and even integers, this isomorphism lends additional justification to the use of the terminology: odd and even.

It may be pointed out that if the total set of functions includes the kth roots of x, then the *composition* defined herein coincides with the definition of composition where ${}_n\mathfrak{F}_k \circ {}_n\mathfrak{F}_s$ is taken as the set of all functions $f[g(x)]$ which arise from the composition of functions in ${}_n\mathfrak{F}_k$ and ${}_n\mathfrak{F}_s$, where $f(x) \in {}_n\mathfrak{F}_k$ and $g(x) \in {}_n\mathfrak{F}_s$.

References

1. Byerly, William E., Fourier's Series and Spherical, Cylindrical, and Ellipsoidal Harmonics, 1893, pp. 46–47.

2. Van der Waerden, B. L., Modern Algebra, 1949, pp. 111–115.

COLLEGE MATHEMATICS FOR NON-SCIENCE STUDENTS

A report of the Special Committee on College Mathematics for Non-Science Students, a Subcommittee of the California Committee for the Study of Education.

This committee has undertaken a study of College Mathematics for General Education, defined as that mathematics which is studied in the freshman or sophomore year as part of a liberal education. It is not a course planned for engineers, professional mathematicians, physical or biological scientists, or even for the modern social scientist who needs specialized mathematics peculiar to his field of research.

I. Current Practices in the Field of College Mathematics for General Education

In order to gain a knowledge of current practices in this field as to curricula, related teaching problems and their solutions, questionnaires were sent to about 250 representative colleges throughout United States and Canada.

Of 110 colleges making replies on the questionnaires, 67 were offering special courses in mathematics for non-science majors, 16 expressed a desire to give such courses, although at present they did not do so. The remaining 27 implied that the usual sequence of mathematics courses was adequate for all of their students.

Most colleges reported that the students who took these special courses had two years of high school mathematics, but in many schools the students had less preparatory mathematics and in a few they had more.

Thirty-three different textbooks were being used, and in six colleges instructors used their own manuscripts. The books used most frequently by those replying on the questionnaires were the textbooks by Richardson; Newsom and Eves; Leonhardy; Cooley, Gans, Kline and Wahlert; Trimble, Peck, Bolser; Jones; Griffin. Relatively few instructors were completely satisfied with any available text indicating that there is essentially no agreement on the topics to be included in a course in mathematics for general education.

Colleges devoted different amounts of time and credit to the course, varying from two to ten semester units. The majority offered either a one semester three credit course or a two semester course, three units each semester. These were in about equal numbers. Most of these courses were required, or they met a general college requirement that could also be satisfied in some other way.

The nature of the course varied greatly in different institutions. In some of the large state universities where they were obliged to take students with little or no high school mathematics, the presence of poor students, large classes and inexperienced teachers made their courses chiefly basic skills courses with emphasis on computation and techniques. They were unhappy about the situation and saw the need for improvement.

In most colleges these courses in mathematics for general education used a historical, intuitive and cultural approach with emphasis on ideas rather than on manipulation or computation. There was a wide range of difficulty. It varied from courses planned for students with no high school mathematical preparation to courses on a high level which emphasized modern mathematics and tried to encourage thinking of an original nature.

Problem solving was the chief type of homework for these courses, but essays and oral reports were frequently required. Practically all organized course activity was limited to the classroom, with few field trips.

The questionnaires also sought information concerning the attitudes of the students, their emotional reactions, what topics they liked most and what they liked least. The replies on these questions were difficult to tabulate, but they indicated (1) that a large proportion of the students who took these courses had had some unfortunate mathematical experience which frequently left them with a negative attitude toward mathematics; and (2) that there is some agreement on topics which inspire enthusiasm and those which antagonize.

The following were mentioned as experiences which caused most enthusiasm: applications; bizarre, new or unusual results; modern implications; problems which can be visualized; problems which the student can solve; challenging situations; discovery on one's own; reasoning; elementary logic; number scales; higher dimensions; transfinite numbers; maximum and minimum problems; problems involving rates; learning how to read stated problems; variation problems; percentage; applications in the field of business; annuities; graphs; statistics and probability; starting at the student's level of understanding and enabling him to achieve success.

Among experiences which seemed to antagonize students the following were mentioned: new definitions; difficult ideas with no applications; not obviously useful or necessary abstractions; logic; topology; proofs, especially detailed tricky proofs; abstract geometry; formal algebra; word problems; lengthy computations; having to give reasons; careful reasoning; thinking; routine.

From an overall examination of the replies on the questionnaires the following generalizations may be drawn:

(1) There is not a single course that meets the needs of all students and all instructors. Successful courses are taught at different levels of mathematical difficulty and maturity.

(2) It is just as important to consider the instructor as it is the students in preparing a course.

(3) The choice of topics is not as important for the success of the course as are the attitude and interests of the teacher, and the attitudes and emotions of the students.

II. Textbooks and Literature on Mathematics for General Education

Much has been published in recent years on the subject of mathematics for general education. The committee has compiled a limited annotated bibliog-

raphy of some textbooks and articles from leading journals which are pertinent to the subject of this study. Copies of this bibliography may be obtained by writing to Professor May M. Beenken, Immaculate Heart College, 2021 North Western Avenue, Los Angeles 27, California.

III. Committee Recommendations

The committee does not recommend a single course to meet the needs of all students and all instructors. Successful courses may be constructed for students of different degrees of mathematical maturity.

The following are guiding principles to consider in constructing a course in mathematics for general education. It should include mathematics that is useful in the home, business and everyday occupations, mathematics which the student will use in other college courses and the mathematics which throws light on the culture of our society. The course should be taught with an adult approach, from a mature and reasonable point of view.

Emphasis in this course will be on critical thinking and understanding rather than technique. But since real understanding must come through application of principles, it must be a course in mathematics and not just about mathematics. Appeal will be made through the beauty of mathematics as well as its utility.

Students who take mathematics for general education are frequently those who have emotional blocks toward the study of mathematics. It will then be not so much the content as the attitude and teaching approach that must distinguish this course from other courses in mathematics. We must meet students where they are and lead them along the path of mathematics as a meaningful aid in precise thinking. This course requires a teacher who is interested in helping students to overcome their blocks and to acquire a liking for mathematics and a feeling of confidence in their ability to do mathematics. The teacher should have a thorough knowledge of mathematical theory and a wide acquaintance with applications to the arts and sciences. He should be well versed in the history of mathematics and its cultural significance. Further qualifications of the successful teacher are patience and an understanding of the learning process.

Beyond such essential topics as the number system, operations with numbers, arithmetic of measurement, functions, graphs, equations and formulas, logical reasoning, the selection of topics should be influenced by the desires and abilities of the students, the interests of the teacher, and the amount of available time. Other topics which may be included are introduction to the calculus, probability and statistics, number theory and some elementary aspects of modern mathematics. In this work the objective will be more an attitude or point of view than a definite amount of knowledge or skill. Some history of mathematics, of important mathematical concepts and the role of mathematics in a world of scientific achievement should be interspersed throughout the entire course.

Upon completion of the course the student should be more critical and exact in his thinking, be better able to meet quantitative situations that arise in everyday living, and he should have a wholesome respect for the power and beauty of

mathematics as a living subject that is constantly growing to meet the needs of an advancing civilization.

A standard course in mathematics for general education should carry three units of credit for each of two semesters. Some colleges may wish to devote more time and credit to this course; others may be restricted to less. Since the success of the course in any college requires the support of all members of the faculty and administration, a major effort should be made to enlist their cooperation through an adequate interpretation of the course to them.

IV. Universal Mathematics

A committee of the Mathematical Association of America on the Undergraduate Mathematical Program has made a thorough study, and advocates a universal freshman course. A first effort at writing a text for such a course has been published as *Universal Mathematics* by the 1954 Summer Writing Group of the University of Kansas.

Universal Mathematics requires a higher level of mathematical preparation than is assumed for the course discussed in this report. This committee urges college mathematics teachers to read the reports of the Committee on the Undergraduate Mathematics Program, and to experiment with the ideas suggested. We agree that students interested in technical mathematics, whether for natural science, social science or the arts, might well take the same course, which should be different from the traditional course. But this is not the class of students for whom a course in mathematics for general education, as defined in this report, is designed.

We feel that it would be inadvisable to give the technically able students, with superior mathematical background of high school mathematics, the same course as average liberal arts students pursuing non-scientific courses, who have had little training in secondary mathematics.

Acknowledgments

This committee wishes to give special recognition to Professor Paul B. Johnson who was responsible for constructing the questionnaire and analyzing the replies; also to Professor Paul A. White who served as chairman of the committee for the year, 1954–55, leaving at the end of the academic year to spend a sabbatical leave in Europe. He relinquished the chairmanship to Professor May M. Beenken for the academic year, 1955–56.

This report was presented April 20, 1956 by Committee:

MAY M. BEENKEN, Immaculate Heart College, Los Angeles
CHARLES J. A. HALBERG, JR., University of California, Riverside
ROBERT B. HERRERA, Los Angeles City College, Los Angeles
PAUL B. JOHNSON, Occidental College, Los Angeles
PAUL A. WHITE, University of Southern California, Los Angeles

MATHEMATICAL NOTES

EDITED BY F. A. FICKEN, University of Tennessee

Because of the large number of papers on hand, consideration of new papers for this department has been temporarily suspended.

ON WALLIS' FORMULA

JOHN GURLAND, Iowa State College

1. Introduction. It is well known that the number π satisfies the following inequalities

$$\frac{2}{2k+1}\left[\frac{(2k)!!}{(2k-1)!!}\right]^2 < \pi < \frac{1}{k}\left[\frac{(2k)!!}{(2k-1)!!}\right]^2, \qquad k = 1, 2, \cdots, \tag{1}$$

where

$$(2k)!! = 2\cdot 4\cdot \cdots \cdot (2k-2)(2k)$$
$$(2k-1)!! = 1\cdot 3\cdot 5 \cdots (2k-3)(2k-1).$$

This result is due to Wallis (*cf.* de la Vallée Poussin [1]).

It is interesting that a basic theorem in mathematical statistics concerning unbiased estimators with minimum variance leads to the following inequalities which yield a closer approximation to π than that afforded by (1), namely

$$\frac{4k+3}{(2k+1)^2}\left[\frac{(2k)!!}{(2k-1)!!}\right]^2 < \pi < \frac{4}{4k+1}\left[\frac{(2k)!!}{(2k-1)!!}\right]^2, \qquad k = 1, 2, \cdots. \tag{2}$$

A statement and a sketch of the proof of this theorem is given below in section 2. Following this, a simple application is given in section 3 which leads to the result (2). In addition to providing this result, the method may be of some interest in itself.

2. A theorem in mathematical statistics. Although a more general formulation of this theorem is possible (*cf.* Cramér [2], Lehmann [3]), the following form of the theorem will be adequate for the purpose considered. The notation $Ef(X)$ denotes, as usual, the mathematical expectation of $f(X)$.

THEOREM. *Let $(X_1, X_2, \cdots, X_n)$ be a random sample from a population with probability density $p(x|\theta)$, where the unknown parameter θ is a number in some open interval I. Suppose $T(X_1, X_2, \cdots, X_n)$ is an unbiased estimator of θ, and assume the regularity conditions* (C) *hold. Then the variance σ_T^2 satisfies the inequality*

$$\sigma_T^2 \geqq \frac{1}{nE\left\{\dfrac{\partial \log p(X|\theta)}{\partial\theta}\right\}^2}. \tag{3}$$

The equality can occur if and only if $T(X_1, X_2, \cdots, X_n)$ *is a linear function of*

$$\sum_{i=1}^{n} \frac{\partial \log p(X_i \mid \theta)}{\partial \theta}.$$

The regularity conditions referred to are the following:

$$C\begin{cases} \text{(i)} & \dfrac{\partial \log p(x \mid \theta)}{\partial \theta} \text{ exists for all } x, \text{ and all } \theta \in I \\ \text{(ii)} & \displaystyle\int_{-\infty}^{\infty} \frac{\partial p(x \mid \theta)}{\partial \theta}\, dx = 0 \\ \text{(iii)} & \displaystyle\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} T(x_1, x_2, \cdots, x_n) \sum_{i=1}^{n} \frac{\partial \log p(x_i \mid \theta)}{\partial \theta}\, dx_1 \cdots dx_n = 1 \\ \text{(iv)} & \displaystyle E\left\{\frac{\partial \log p(X \mid \theta)}{\partial \theta}\right\}^2 > 0. \end{cases}$$

Proof. The method of proof is briefly indicated here. Let

$$\phi(X_1, X_2, \cdots, X_n) = \sum_{i=1}^{n} \frac{\partial \log p(X_i \mid \theta)}{\partial \theta}.$$

Then the regularity conditions (C) imply that $E\phi(X_1, X_2, \cdots, X_n) = 0$ and $ET(X_1, \cdots, X_n)\phi(X_1, \cdots, X_n) = 1$. Since the correlation coefficient between T and ϕ must lie in the interval from -1 to 1, it follows that

$$E(T - ET)^2 E\phi^2 \geqq ET\phi.$$

This yields the required result.

3. Proof of (2). Consider a random sample $(X_1, X_2, \cdots, X_n)$ from a normal population with mean 0 and variance σ^2. Then

$$p(x \mid \sigma) = \frac{1}{\sigma\sqrt{2\pi}}\, e^{-x^2/2\sigma^2}.$$

Now $(1/\sigma^2)\sum_{i=1}^{n} X_i^2$ has a χ^2 distribution with n degrees of freedom; hence $(1/\sigma)\sqrt{\sum_1^n X_i^2}$ has probability density

$$\frac{1}{2^{(n/2)-1}\Gamma\left(\dfrac{n}{2}\right)}\, v^{n-1}e^{-v^2/2}, \qquad v > 0,$$

the first moment of which is

$$a = \frac{\sqrt{2}\,\Gamma\left(\dfrac{n+1}{2}\right)}{\Gamma\left(\dfrac{n}{2}\right)}.$$

Consequently

$$T = \frac{1}{a}\sqrt{\sum_1^n X_i^2}$$

is an unbiased estimator σ. It is easily verified that the variance of T is given by

$$\sigma_T^2 = \sigma^2\left(\frac{n}{a^2} - 1\right).$$

Further, since

$$\frac{\partial \log p(x \mid \sigma)}{\partial \sigma} = -\frac{1}{\sigma} + \frac{x^2}{\sigma^3},$$

the lower bound for σ_T^2 given in (2) is $\sigma^2/2n$. However, since

$$\phi(X_1, X_2, \cdots, X_n) = -\frac{n}{\sigma} + \frac{1}{\sigma^3}\sum_1^n X_i^2,$$

it is clear that T is not a linear function of ϕ; hence the bound in (3) is not attained by σ_T^2. Thus

$$\sigma^2\left[\frac{n}{a^2} - 1\right] > \frac{\sigma^2}{2n},$$

which reduces to

$$\left[\frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n}{2}\right)}\right]^2 < \frac{n^2}{2n+1}, \qquad n = 1, 2, \cdots. \tag{4}$$

For $n = 2k$, the inequality (4) may be written as

$$\pi < \left[\frac{(2k)!!}{(2k-1)!!}\right]^2 \frac{4}{4k+1}, \qquad k = 1, 2, \cdots$$

while for $n = 2k+1$ it may be written as

$$\frac{4k+3}{(2k+1)^2}\left[\frac{(2k!!)}{(2k-1)!!}\right]^2 < \pi, \qquad k = 1, 2, \cdots.$$

Combining these two results yields the inequalities in (2).

References

1. Ch.-J. de la Vallée Poussin, Cours d'Analyse Infinitésimale, Gauthier-Villars, Paris.
2. H. Cramér, Mathematical Methods of Statistics, Princeton University Press, 1946.
3. E. L. Lehmann, Notes on the theory of estimation, Mimeographed Notes.

A MINIMUM SOLUTION OF A DIOPHANTINE EQUATION

R. J. LEVIT, Massachusetts Institute of Technology

Let a, b, and c be given integers, a and b relatively prime, and $ab \neq 0$. The traditional procedure in solving the diophantine equation,

$$ax + by = c, \tag{1}$$

namely, determining a solution of

$$a\xi + b\eta = 1 \tag{2}$$

and multiplying it by c, generally leads to excessively large numerical values for x and y and consequently to an unnecessarily cumbersome form for the general solution. For example, in the case of the equation,

$$94x + 53y = 77, \tag{3}$$

which is satisfied by $x = -2$, $y = 5$, it yields the solution, $x = 1694$, $y = -3003$.

On the other hand, there is a familiar elementary method of solving (1) independently of (2) which does not have this defect. We recall this method by applying it to the example (3). Solving for y, we obtain $y = (77 - 94x)/53 = 1 - 2x + (24 + 12x)/53$. Then, since x and y are integers, so is $w = (24 + 12x)/53$; and w satisfies another diophantine equation, $53w - 12x = 24$, of the form (1) but with smaller coefficients than (3). Repeating the process by solving the latter equation for x, we arrive at a homogeneous equation, $12v - 5w = 0$, which has the obvious solution $w = v = 0$. Then we find from the preceding equation that $x = -2$ and hence $y = 5$. This we call the *definitely least solution* in the sense that $|x|$ and $|y|$ both attain their least possible values. It is the purpose of this note to show that with a slight modification, which incidentally shortens the computation, this method always leads to the definitely least solution when it exists and, when it does not, to a solution in which, with the notation arranged so that $|a| \geqq |b|$, $|x|$ assumes its minimum value. This we call a *minimum solution with respect to* x. We also give conditions for the existence of the definitely least solution.

It is not necessary to write out and solve explicitly the successive equations as was done in the example. The process can be reduced to an algorithm. In this form it is a generalization to arbitrary c of a highly efficient algorithm suggested originally by Gauss for the solution of (2), which, as D. H. Lehmer has pointed out,* requires only half as much labor as the usual continued fraction procedure. Setting $b_0 = a$, $b_1 = b$, $c_0 = c$ for uniformity of notation, we perform successively for $i = 1, 2, \cdots, n$ the two sequences of divisions with numerically least remainders,

$$b_{i-1} = q_i b_i + b_{i+1}, \qquad |b_{i+1}| \leqq \tfrac{1}{2}|b_i|, \tag{4}$$

$$c_{i-1} = g_i b_i + c_i, \qquad |c_i| \leqq \tfrac{1}{2}|b_i|. \tag{5}$$

* A note on the linear diophantine equation, this MONTHLY, vol. 48, 1941, pp. 240–246.

The value $i=n$ at which the divisions terminate is uniquely determined by the conditions,

$$0 < |c_i| < \tfrac{1}{2}|b_i| \qquad (i = 1, 2, \cdots, n-1) \tag{6}$$

and either

$$c_n = 0, \quad b_n \neq 0, \text{ or} \tag{7a}$$

$$c_n = \pm \tfrac{1}{2}b_n \neq 0. \tag{7b}$$

That (6) and (7) will hold for some integer n follows from the fact that the sequence (4) would constitute a Euclidean algorithm for the greatest common divisor (b_0, b_1) if continued until a zero remainder b_{m+1} was reached. Then, $b_m=(b_0, b_1)=(a, b)=1$, and hence by (5) for $i=m$, $c_m=0$, so that (6) and either (7a) or (7b) are satisfied by some positive integer $n \leqq m$. This brings us to our algorithm.

THEOREM 1. *From the q_i, b_i, g_i determined in the divisions* (4) *and* (5) *let integers x_i be computed as follows: x_n and x_{n+1} are defined by either*

$$x_n = x_{n+1} = 0, \text{ or} \tag{8a}$$

$$x_n = \pm \tfrac{1}{2}b_n, \quad x_{n+1} = \mp [\tfrac{1}{2}b_{n+1}] \tag{8b}$$

accordingly as (7a) *or* (7b) *holds respectively. The upper signs are to be used in* (8b) *if the upper sign was chosen in* (7b); *otherwise the lower signs are to be used. Then $x_{n-1}, x_{n-2}, \cdots, x_1, x_0$ are obtained successively by the recurrence relation,*

$$x_{i-1} = g_i - q_i x_i + \tfrac{1}{2}x_{i+1}. \tag{9}$$

Then

$$x = x_1, \quad y = x_0 \tag{10}$$

is a solution of the diophantine equation,

$$b_0 x + b_1 y = c_0, \qquad |b_0| \geqq |b_1| > 0, \ (b_0, b_1) = 1. \tag{11}$$

More generally, we show that

$$b_j x_{j+1} + b_{j+1} x_j = c_j \qquad (j = 0, 1, \cdots, n). \tag{12}$$

When (8a) applies, (12) is immediate for $j=n$. (8b) can apply only when b_n is even and hence only when b_{n+1} is odd so that $x_{n+1}=\mp\frac{1}{2}(b_{n+1}-1)$, from which we can again verify (12) for $j=n$ by direct substitution. Now to complete the proof by finite descending induction, suppose (12) holds for a value $j=i>0$. Then $x_{i+1}=(c_i-b_{i+1}x_i)/b_i$ so that from (9), (5) and (4)

$$x_{i-1} = g_i - q_i x_i + \frac{c_i - b_{i+1}x_i}{b_i} = \frac{g_i b_i + c_i - (q_i b_i + b_{i+1})x_i}{b_i} = \frac{c_{i-1} - b_{i-1}x_i}{b_i},$$

and hence (12) holds for $j=i-1$ also. Thus (12) is established for $j=n, n-1, \cdots, 0$; and the case $j=0$ is precisely our theorem.

To establish the minimum property of the solution (10) we state the following lemma, which is readily proved from the usual form of the general solution of (1).

LEMMA. *A solution (X, Y) of* (1) *is a minimum solution with respect to x (y) if and only if $|X| \leqq \frac{1}{2}|b|$ ($|Y| \leqq \frac{1}{2}|a|$), and is the only such solution except when the equality holds. It is the definitely least solution if and only if both $|X| \leqq \frac{1}{2}|b|$ and $|Y| \leqq \frac{1}{2}|a|$.*

THEOREM 2. *Let the integers x_i be computed as in Theorem* 1 *with the stipulation that the ambiguous sign in* (7b) *is to be chosen positive if b_{n-1} and c_{n-1} have the same sign and negative if they have opposite sign. Then $x=x_1$, $y=x_0$ is the minimum solution of the diophantine equation* (11) *with respect to x.*

Proof. We first show that for $n>1$

$$|x_j| < \tfrac{1}{2}|b_j| \qquad (j = 1, 2, \cdots, n-1). \tag{13}$$

When (8a) applies, $|b_n x_{n-1}| = |b_{n-1}x_n + b_n x_{n-1}| = |c_{n-1}| < \frac{1}{2}|b_{n-1}|$ by (12), (6) and hence, since $b_n \neq 0$ by (7a), $|x_{n-1}| < \frac{1}{2}|b_{n-1}|$. When (8b) applies and the sign in (7b) is chosen as stipulated in the hypothesis, $x_n = \frac{1}{2}|b_n|$ sgn $(b_{n-1}c_{n-1})$ so that by (12)

$$\begin{aligned} |x_{n-1}| &= \left|\frac{c_{n-1} - b_{n-1}x_n}{b_n}\right| = \left|\frac{c_{n-1} - \frac{1}{2}|b_{n-1}b_n| \text{ sgn } c_{n-1}}{b_n}\right| \\ &= \left|\frac{|c_{n-1}| - \frac{1}{2}|b_{n-1}b_n|}{b_n}\right| = \left|\left|\frac{c_{n-1}}{b_n}\right| - \frac{1}{2}|b_{n-1}|\right|. \end{aligned} \tag{14}$$

But $|c_{n-1}/b_n| \leqq |c_{n-1}| < \frac{1}{2}|b_{n-1}|$ by (6) and hence again $|x_{n-1}| < \frac{1}{2}|b_{n-1}|$. Thus (13) is established for $j=n-1$. In the case $n=2$ the proof of (13) is already complete. For $n>2$ the proof is by descending induction; so we suppose (13) holds for any value $j=k>1$. Then, since b_k and x_k are integers, $2|x_k| \leqq |b_k| - 1$. Hence by (12) and (6)

$$\begin{aligned} |x_{k-1}| &= \left|\frac{c_{k-1} - b_{k-1}x_k}{b_k}\right| \leqq \left|\frac{c_{k-1}}{b_k}\right| + \left|\frac{b_{k-1}x_k}{b_k}\right| \\ &< \left|\frac{b_{k-1}}{2b_k}\right| + \left|\frac{b_{k-1}(|b_k| - 1)}{2b_k}\right| = \frac{1}{2}|b_{k-1}|; \end{aligned}$$

so (13) holds for $j=k-1$ also. Thus (13) is established for $j=n-1, n-2, \cdots, 1$ provided $n>1$. When $n=1$, $x_1=0$ or $x_1 = \pm\frac{1}{2}b_1$ by (8a) or (8b). Combining this with the case $j=1$ of (13), we have

$$|x_1| < \tfrac{1}{2}|b_1| \text{ when } n > 1; \qquad |x_1| \leqq \tfrac{1}{2}|b_1| \text{ when } n = 1. \tag{15}$$

Hence by the lemma (x_1, x_0) is a minimum solution of (11) with respect to x.

We note that by the lemma (x_1, x_0) is the only minimum solution of (11) with respect to x except when $x_1 = \pm\frac{1}{2}b_1$. Then there is just one other such solution, say (x_1', x_0'); and, since by the lemma $|x_1'| = \frac{1}{2}|b_1|$ also, we must have $x_1' = -x_1$. In view of (15) this case can occur only when $n = 1$. Then (14) becomes $|x_0| = \left||c_0/b_1| - \frac{1}{2}|b_0|\right|$, while in like manner $|x_0'| = |c_0 - b_0x_1'|/|b_1| = |c_0 + b_0x_1|/|b_1| = |c_0/b_1| + \frac{1}{2}|b_0|$; so $|x_0'| > |x_0|$. This proves the corollary,

COROLLARY 1. *Either (x_1, x_0) is the only minimum solution of* (11) *with respect to x or there is just one other, and then (x_1, x_0) is the one in which $|y|$ is smaller.*

Since the definitely least solution of (11), if it exists, is *a fortiori* a minimum solution with respect to x and is the one in which $|y|$ is least, we then have

COROLLARY 2. *(x_1, x_0) is the definitely least solution of* (11) *whenever such a solution exists.*

The following example illustrates the operation of the algorithm with the termination (7b) in conjunction with the convention for choosing the ambiguous sign:

(16) $$20243x + 9153y = 69084$$

i	b_i	q_i	c_i	g_i	x_i
0	20243		69084		−10104
1	9153	2	5013	7	4572
2	1937	5	−798	3	−967
3	−532	−4	−266	1	−266
4	−191				96

Since $|c_3| = \frac{1}{2}|b_3|$, $n = 3$. b_2 and c_2 have opposite sign; so the quotient g_3 is chosen to give a negative remainder c_3. Then, since $c_3 = +\frac{1}{2}b_3$, we use the upper signs in (8b) to obtain $x_3 = \frac{1}{2}b_3 = -266$, $x_4 = -[\frac{1}{2}b_4] = -[-191/2] = 96$. The rest of the last column is then calculated upward by means of (9) giving finally $x = x_1 = 4572$, $y = x_0 = -10104$, the definitely least solution. The traditional method yields the solution $x = 46{,}355{,}364$, $y = -102{,}520{,}656$.

We conclude by establishing conditions for the existence of a definitely least solution (d.l.s.) of (1). There is no loss of generality if we take b odd, $a > 0$, $b > 0$, $c \geqq 0$.

THEOREM 3. *Let $a > 0$, $b > 0$, $c \geqq 0$. There are integers L and R depending only on a and b such that the diophantine equation* (1) *has a* d.l.s. *for every $c < L$, for no $c > R$, and for exactly half the values of c in the interval $L \leqq c \leqq R$. There is a* d.l.s. *for one but not the other of each pair of values $c = C$ and $c = C'$ such that $C + C' = L + R$. In particular, there is a* d.l.s. *for $c = R$ but not for $c = L$. With the notation arranged so that b is odd the values of L and R are given explicitly by*

(17) *a odd:* $L = \frac{1}{2}(a + b)$, $R = ab - \frac{1}{2}(a + b)$;

(18) *a even:* $L = \frac{1}{2}a + b$, $R = ab - \frac{1}{2}a$.

Proof. Let a and b be odd and let (X, Y) be a solution of (1) for the value $c=C$. Then $a(-X)+b(a-Y)=ab-(aX+bY)=ab-C$; *i.e.*, $(-X, a-Y)$ is a solution of

$$ax+by=C'=L+R-C \tag{19}$$

with L and R given by (17). Now, if (X, Y) is the d.l.s. of (1) for $c=C$, by the lemma (to Theorem 2) $|X|<\frac{1}{2}b$, $|Y|<\frac{1}{2}a$. Hence also $|-X|<\frac{1}{2}b$ but $|a-Y|>\frac{1}{2}a$ so that by the lemma $(-X, a-Y)$ is the (unique) minimum solution with respect to x of (19) but not the d.l.s. However, when the d.l.s. exists, it is necessarily also a minimum solution with respect to x. Hence we conclude that (19) has no d.l.s. Conversely, suppose (1) has no d.l.s. for a value $c=C$ such that $0\leqq C\leqq L+R$ but that the solution (X, Y) is a minimum with respect to x so that $|X|<\frac{1}{2}b$. Then $Y>\frac{1}{2}a$; for otherwise, since (X, Y) is not a d.l.s., $Y<-\frac{1}{2}a$, and it would follow that $C=aX+bY<a\cdot\frac{1}{2}b-b\cdot\frac{1}{2}a=0$ contrary to hypothesis. At the same time $Y<3a/2$, since, if $Y\geqq 3a/2$, we would have $C>a(-\frac{1}{2}b)+b(3a/2)=ab=L+R$ as given by (17). Accordingly $\frac{1}{2}a<Y<3a/2$ and hence $-\frac{1}{2}a=a-3a/2<a-Y<a-\frac{1}{2}a=\frac{1}{2}a$, or $|a-Y|<\frac{1}{2}a$; and, since $|-X|<\frac{1}{2}b$, the solution $(-X, a-Y)$ of (19) is a d.l.s. For a even we readily verify that, if (X, Y) is a solution of (1) for $c=C$, $(-X, a+1-Y)$ is a solution of (19) with L and R given by (18). We can then proceed as before to show that, if $0\leqq C\leqq L+R$, one but not both of the equations (1) (with $c=C$) and (19) has a d.l.s. Next, if (1) has the d.l.s., (X, Y) and a and b are odd, $|X|\leqq\frac{1}{2}(b-1)$, $|Y|\leqq\frac{1}{2}(a-1)$, and $c\leqq a|X|+b|Y|\leqq a\cdot\frac{1}{2}(b-1)+b\cdot\frac{1}{2}(a-1)=R$ as given by (17). On the other hand, if a is even, $|Y|\leqq\frac{1}{2}a$ and then $c\leqq a\cdot\frac{1}{2}(b-1)+b\cdot\frac{1}{2}a=R$ as given by (18). In either case (1) can have no d.l.s. for $c>R$, but for $c=R$ it does have the d.l.s., $x=\frac{1}{2}(b-1)$, $y=\frac{1}{2}(a-1)$(a odd), $y=\frac{1}{2}a$(a even). Finally, since, as we have just seen, there is no d.l.s. for any value $c=C$ such that $R<C\leqq L+R$, there is a d.l.s. for every value $c=C'=L+R-C$, where $0\leqq C'<L$. However, because there is a d.l.s. for $c=C=R$, there is none for $c=C'=L$.

Thus, referring to the example (3), $94x+53y=c$ has a d.l.s. for every integer $c<100$ but for no $c>4935$, there being a d.l.s. for $c=4935$ but not for $c=100$. Since from the example (16) $20243x+9153y=c$ has a d.l.s. for $c=69084$, it has none for $c=20243\cdot 9153-69084=185{,}215{,}095$. There is a d.l.s. for every integer $c<14698$ but for no $c>185{,}269{,}481$.

It is natural to ask how our results would be affected if in the divisions (4) and (5) we used least positive remainders instead of least absolute remainders. The following is easily shown:

THEOREM 4. *Let $b_0>b_1>0$. Replace the inequalities in* (4) *and* (5) *by*

$$0\leqq b_{i+1}<b_i, \qquad 0\leqq c_i<b_i$$

respectively, terminating the sequence of divisions with (7a). *Determine the integers x_i by* (8a) *and* (9). *Then $x=x_1$, $y=x_0$ is a solution of* (11), *and $|x_1|<b_1$.*

The equation $11x+7y=6$ has the solution $(-2, 4)$ while in this case $x_1=5$, $x_0=-7$; *i.e.*, (x_1, x_0) is not necessarily a minimum solution of (11) with respect to x. However, since (11) has only two solutions with $|x|<b_1$, it follows from Theorem 4 that there is at most one solution such that $|x|<|x_1|$.

A SINGULAR INTEGRAL EQUATION CONTAINING A PARAMETER

J. R. Hatcher, Fisk University

This note concerns the inversion of the singular equation

$$(\pi i)^{-1}\int_L \cos\lambda(t-t_0)g(t)dt/(t-t_0) = f(t_0), \qquad t_0 \in L, \tag{1}$$

where L (consisting of a finite number of smooth non-intersecting contours of finite length $L_0, L_1, \cdots, L_n$ with L_0 containing all the rest) is the boundary of a plane connected region, the functions $f(t)$ and $g(t)$ are subject to the Hölder condition on L, and λ is a parameter.

The well-known solution of (1) for $\lambda=0$, *i.e.*,

$$g(t_0) = (\pi i)^{-1}\int_L f(t)dt/(t-t_0), \tag{2}$$

indicates the possibility of a general solution of the form

$$g(t_0) = (\pi i)^{-1}\int_L G[\lambda(t-t_0)]f(t)dt/(t-t_0), \tag{3}$$

where $G[\lambda(t-t_0)]$ is of Hölder type with respect to t, t_0, and $G(0)=1$.

Thus, replacing $g(t)$ in (1) by means of (3) and applying the Poincaré-Bertrand transformation formula [1, p. 56]

$$\int_L \frac{dt}{t-t_0}\int_L \frac{K(t,\xi)}{\xi-t}d\xi = -\pi^2K(t_0,t_0) + \int_L d\xi\int_L \frac{K(t,\xi)}{(t-t_0)(\xi-t)}dt, \tag{4}$$

it is seen that the function $G(t)$ must satisfy the following conditions:

$$\int_L f(\xi)d\xi\int_L \frac{\cos\lambda(t-t_0)G[\lambda(\xi-t)]}{(t-t_0)(\xi-t)}dt = 0, \qquad G(0)=1. \tag{5}$$

But an expansion of $[(t-t_0)(\xi-t)]^{-1}$ in partial fractions immediately reveals $G(t)=\cos t$. For

$$\int_L \frac{\cos\lambda(t-t_0)G[\lambda(\xi-t)]}{(t-t_0)(\xi-t)}\,dt$$

$$= \frac{1}{\xi-t_0}\int_L \left\{\frac{1}{t-t_0}-\frac{1}{t-\xi}\right\}\cos\lambda(t-t_0)G[\lambda(\xi-t)]\,dt \tag{6}$$

$$= \frac{\pi i}{\xi-t_0}\left\{G[\lambda(\xi-t_0)]-\cos\lambda(\xi-t_0)\right\}$$

provided $G[\lambda(\xi-Z)]$ is analytic as a function of Z, where Z belongs to the region bounded by L. Therefore $G(t)=\cos t$ satisfies (5), and hence the relations

$$(\pi i)^{-1}\int_L \cos\lambda(t-t_0)g(t)dt/(t-t_0) = f(t_0) \quad \text{and} \tag{7}$$

$$(\pi i)^{-1}\int_L \cos\lambda(t-t_0)f(t)dt/(t-t_0) = g(t_0) \quad \text{are reciprocal.} \tag{8}$$

Further, it is easily seen that similar relations hold for other special functions $K[\lambda(t-t_0)]$ with $K(0)=1$, *e.g.*, $\cosh\lambda(t-t_0)$.

Reference

1. N. I. Muskhelishvili, Singular Integral Equations, Translation from the Russian edited by J. R. M. Radok, P. Noordhoff, N. V. Groningen-Holland, 1953.

THE ROW-SUMS OF THE INVERSE MATRIX, II

ALBERT WILANSKY, Lehigh University

It was remarked in I (this MONTHLY, vol. 58, 1951, pp. 614–615) that an obvious fact about finite matrices failed to hold for infinite matrices, namely, if the row-sums of a matrix A are s those of its inverse are $1/s$. Let us state this as (*); *if* $BA=I$, $\sum_k a_{nk}=s$ *for each* n; *then* $\sum_k b_{nk}=1/s$ *for each* n. *Proof* (with $s=1$): $B1=B(A1)=(BA)1=1$, where 1 is the column vector of ones.

Clearly (*) holds for infinite matrices if B is row-finite, or if $\|A\|<\infty$, $\|B\|<\infty$, where $\|A\|=\sup_n \sum_k |a_{nk}|$. That neither of these hypotheses can be dropped follows from an example in I.

In the finite case, $s=0$ implies that B does not exist. We give an example to show that this is false in the infinite case. Specifically, *there exists a matrix* A (which turns out to be row-finite) *with row-sums zero, and a matrix* B *with row-sums one, such that* $BA=I$. (Example 1, below.) As a variant of (*) we state: *if* B *is row-finite and* 1-1 (*i.e.* $Bx=0$ *implies* $x=0$), *if* $BA=I$ *and* $\sum_k b_{nk}=s$, *then* $\sum_k a_{nk}=1/s$ (and $s\neq 0$).

Proof (with $s=1$): $B(1-A1)=1-B(A1)=1-(BA)1=0$.

We may replace the hypothesis of row-finiteness by a boundedness assump-

tion as in (*). *Neither "row-finite" nor "1-1" may be omitted.* That "1-1" may not be omitted follows from the last paragraph but one of I. That "row-finite" may not be omitted follows from Example 1, below, in which it may be checked that B is 1-1 (simpler examples are available without this property).

Finally, Example 1 also verifies the conjecture made in I that (*) need not hold for infinite matrices if it is assumed only that $\sum_k b_{nk}$ exists for each n.

These failures result of course from the non-associative character of matrix multiplication. (See Vermes, Non-associative rings of infinite matrices, *Math. Rev.*, vol. 14, 1953, p. 7.)

Example 1.

$$B = \begin{Vmatrix} 1/2 & 1/4 & 1/8 & 1/16 & 1/32 \cdots \\ 3/4 & 0 & 1/8 & 1/16 & 1/32 \cdots \\ 7/8 & 0 & 0 & 1/16 & 1/32 \cdots \\ 15/16 & 0 & 0 & 0 & 1/32 \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{Vmatrix}$$

$$A = \begin{Vmatrix} 0 & 0 & 0 & 0 & 0 \cdots \\ 4 & -4 & 0 & 0 & 0 \cdots \\ 0 & 8 & -8 & 0 & 0 \cdots \\ 0 & 0 & 16 & -16 & 0 \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{Vmatrix}$$

Reference

A. Wilansky and K. Zeller: Inverses of matrices and matrix transformations, Proc. Amer. Math. Soc., vol. 6, 1955, pp. 414–420.

FUNCTIONS WITH PREASSIGNED DERIVATIVES

E. B. Leach, Case Institute of Technology

The existence of a function of one real variable whose derivatives of all orders have preassigned values at a point has been shown in a simple way by A. Rosenthal.* In this note we give an alternative proof, and make the extension that if the initial values for the derivatives are C^∞ functions of a number of real parameters, the desired function can be chosen to be a C^∞ function of the parameters. This proposition has for a simple corollary the corresponding proposition for functions of several real variables whose partial derivatives are to have preassigned values at a point.

* A. Rosenthal. On functions with infinitely many derivatives, Proc. Am. Math. Soc., vol. 4, pp. 600–602, 1953.

THEOREM 1. *Let* $\{a_k(u)\}$, $k=0, 1, 2, \cdots$ *be a sequence of* C^∞ *functions of the* m*-dimensional real variable* $u=(u_1, \cdots, u_m)$, *each defined for* $u \in U$, *an open set in* R^m. *Then there is a* C^∞ *function* $f(x, u)$ *defined for* $x \in R^1$, *and* $u \in U$, *such that* $\partial^k f(0, u)/\partial x^k = a_k(u)$, $k=0, 1, 2, \cdots$.

Proof. The obvious formal solution is:

$$f(x, u) \sim \sum_{\mu=0}^{\infty} a_\mu(u) x^\mu/\mu!. \tag{1}$$

Let $\phi(x)$ be a function of x satisfying:

(i) $\phi(x)$ is a C^∞ function defined for $x \in R^1$

(ii) $\phi(x)=0$ for $|x| \geqq 1$

(iii) $\phi(0)=1$; $\phi^{(k)}(0)=0$ for $k=1, 2, \cdots$.

Such a function is readily shown to exist. We could take for example:

$$\phi(x) = \begin{cases} \exp\left[-(1-|x|)^{-1} \exp\left(-|x|^{-1}\right)\right], & 0 < |x| < 1 \\ 0, & |x| \geqq 1 \\ 1, & x = 0. \end{cases}$$

We will now modify the formal solution (1) to take the form:

$$f(x, u) = \sum_{\mu=0}^{\infty} a_\mu(u) \frac{x^\mu}{\mu!} \phi(x/\epsilon_\mu). \tag{2}$$

In view of the property (iii) of the function ϕ, it is evident that (2) is again a formal solution to our problem. We shall show that it is possible to choose positive numbers ϵ_μ, so that (2), and the result of differentiating (2) any number of times formally with respect to x and a combination of the variables u, converges uniformly on every compact subset of $R^1 \times U$. By well-known theorems of analysis, this implies that f has the properties asserted in the theorem.

Let us consider the contribution of the μth term of (2) to a formal derivative, p times with respect to a combination of the u variables and q times with respect to x. This will be the product of two factors:

(a) The corresponding pth derivative of $a_\mu(u)$

(b) The qth derivative of $(x^\mu/\mu!)\phi(x/\epsilon_\mu)$.

To find an appropriate bound for the factor (a), let $\{C_k\}$, $k=1, 2, \cdots$ be an expanding sequence of compact subsets of U whose interiors cover U. Then $a_k(u)$ and each of its derivatives is bounded on C_k. Let N_k be a common upper bound on the magnitude of $a_k(u)$, and of its derivatives of orders $\leqq k-1$. Then the factor (a) will be no greater than N_μ, if $\mu \geqq p+1$, $\mu \geqq k$, and $u \in C_k$, since $\{C_k\}$ is an expanding sequence of sets.

To bound (b), let M_k be a common upper bound for the magnitudes of $\phi(x)$, and of its derivatives of orders $\leqq k-1$. The qth derivative of $(x^\mu/\mu!)\phi(x/\epsilon_\mu)$ is the sum of 2^q terms, each of which is the product of an rth derivative of $(x^\mu/\mu!)$ and an sth derivative of $\phi(x/\epsilon_\mu)$, where $r+s=q$. Since $\phi(x/\epsilon_\mu)$ and its derivatives

vanish for $|x| > \epsilon_\mu$, we may assume $|x| \leqq \epsilon_\mu$, and a bound for any one of the terms will be $(\epsilon_\mu)^{\mu-r} \cdot \epsilon_\mu^{-s} M_\mu = \epsilon_\mu^{\mu-q} M_\mu$, the factor ϵ_μ^{-s} arising from the chain rule for differentiating $\phi(x/\epsilon_\mu)$. This bound is valid, provided $\mu \geqq q+1$. Multiplying by the number of terms gives $2^q \epsilon_\mu^{\mu-q} M_\mu$ as a bound for (b), when $\mu \geqq q+1$.

Multiplying the bounds for (a) and (b) together, we obtain $2^q \epsilon_\mu^{\mu-q} M_\mu N_\mu$ as a bound for the contribution of the μth term of (2) to the formal derivative we are considering, provided $u \in C_k$, and $\mu \geqq \sup(p+1, q+1, k)$. Now, if ϵ_μ is chosen so small that $\epsilon_\mu M_\mu N_\mu \leqq 1$, the bound may be replaced by $2^q \epsilon_\mu^{\mu-q-1}$; and if $0 < \epsilon_\mu \leqq 1/2$, the bound may be replaced by $(1/2)^{\mu-2q-1}$, since the exponent of ϵ_μ in the previous bound is non-negative, for $\mu \geqq q+1$. Since the new bound is the general term in a convergent series of positive constants, we have established uniform convergence of the result of differentiating (2) term by term any number of times, for $u \in C_k$ which in turn shows that the function $f(x, u)$ defined by (2) satisfies the conditions of the theorem.

We can extend this result immediately to the case of several real variables:

COROLLARY. *Let $a_{k_1 \cdots k_n}(u)$, $k_1, \cdots, k_n = 0, 1, 2, \cdots$ be a collection of C^∞ functions defined for $u \in U$, as in the theorem. Then there is a C^∞ function $f(x, u)$, where $x = (x_1, \cdots, x_n)$, defined for $x \in R^n$, and $u \in U$, such that for every choice of indices, $k_1, \cdots, k_n$, we have:*

$$\frac{\partial^k f(0, u)}{\partial x_1^{k_1} \cdots \partial x_n^{k_n}} = a_{k_1 \cdots k_n}(u), \qquad k = k_1 + \cdots + k_n.$$

Proof. We use induction on n, the theorem being the case $n = 1$. For $n > 1$, let $x' = (x_1, \cdots, x_{n-1})$; and by the induction hypothesis, there exists, for each integer k_n, a C^∞ function $f_{k_n}(x', u)$, satisfying:

$$\frac{\partial^k f_{k_n}(0, u)}{\partial x_1^{k_1} \cdots \partial x_{n-1}^{k_{n-1}}} = a_{k_1 \cdots k_n}(u), \qquad k = k_1 + \cdots + k_{n-1}.$$

Letting $x'' = x_n$, there is now, by the theorem for the case $n = 1$ a C^∞ function $f(x', x'', u)$, such that for each value of k_n:

$$\frac{\partial^{k_n} f(x', 0, u)}{\partial x''^{k_n}} = f_{k_n}(x' u).$$

It is immediately verified that under the identification $(x', x'') = (x_1, \cdots, x_n)$, the function f has the desired properties.

CLASSROOM NOTES

EDITED BY G. B. THOMAS, Massachusetts Institute of Technology

Because of the large numbers of papers on hand, consideration of new papers for this department has been temporarily suspended.

THE CUBIC EQUATION AND THE MANNHEIM SLIDE RULE

H. A. ARNOLD, University of California, Davis

The graphical discussion of Klamkin in [1], if slightly modified, leads to an unusually simple slide rule solution of the cubic, in which, except for a preliminary cube root, the only adjustment made is with the cursor (hairline). The cubic, with real coefficients,

$$z^3 + a_1z^2 + a_2z + a_3 = 0, \qquad a_3 \neq 0, \tag{1}$$

can be reduced by the substitution $z = w - a_1/3$ to the form

$$w^3 + aw + b = 0, \qquad b \neq 0, \tag{2}$$

where

$$\begin{aligned} a &= (3a_2 - a_1^2)/3 \\ b &= (2a_1^3 - 9a_1a_2 + 27a_3)/27. \end{aligned} \tag{3}$$

Setting $w = b^{1/3}x$, equation (2) reduces to

$$x^3 - Sx + 1 = 0, \qquad S = -(a/b)b^{1/3}, \tag{4}$$

$$x^2 + 1/x = S. \tag{4'}$$

$|b|^{1/3}$ is found with or without the K-scale using the slide-rule instruction.

Using the standard relations between the roots and coefficients of a cubic, see [2, p. 64], equation (4) has exactly one negative root. It has no positive (hence two conjugate imaginary) roots, two coincident or two distinct positive roots according as the discriminant $\Delta = -4a^3 - 27b^2$ is <0, $=0$, >0; that is, as S is less than, equal to, or greater than $3/4^{1/3} = 1.890$. See [2, pp. 95–97].

Close the slide rule, with the A, C' and D-scales visible and their ends aligned. (If the tongue is upside down, the C-scale serves as a C'-scale). The hairline (in any position) now lies over: some number x on the D-scale; $1/x$ on the C'-scale; x^2 on the A-scale. This remark used with equation (4') justifies Steps I and II, and fixes the position of the decimal point on each scale.

Step I. The negative root, $-v$. For all S. Set the hairline so that

$$A\text{-reading } \textit{minus } C'\text{-reading} = S. \tag{5}$$

The negative of the D-reading is the root $-v$. If S is a negative number, multiply equation (5) by -1.

Step II. The positive roots. $S \geqq 3/4^{1/3} = 1.890$. Set the hairline so that

(6) $$A\text{-reading } \textit{plus } C'\text{-reading} = S.$$

The D-reading is a positive root. Two such settings, or one, or no such setting is possible. The position of the decimal point may be different on a given scale for each setting.

After Step I, one may divide equation (4) by $x-(-v)$ and solve the resulting quadratic for the remaining roots x_2, x_3

(7) $$x^2 - vx + 1/v = 0$$

(8) $$x_2,\ x_3 = \tfrac{1}{2}(v \pm \sqrt{v^2 - 4/v}) \text{ if } v \geqq 4^{1/3} = 1.587+,$$
$$x_2,\ x_3 = \tfrac{1}{2}(v \pm i\sqrt{-v^2 + 4/v}) \text{ if } v < 4^{1/3}.$$

This yields any imaginary roots and checks Step II for real roots. As a final check, the sum of the roots of (4′) is zero.

An example is equation (4), where $S=3.5$. In Step I, the readings for A, C', D are 4, .5, 2 respectively, so $x_1=-2$; in Step II the two sets of respective readings are 2.91, 0.586, *1.706* and 0.086, 3.41, *0.293*. Consequently, $x_2=1.706$, and $x_3=0.293$.

References

1. M. S. Klamkin, A geometric determination of the nature of the roots of the cubic, biquadratic and quintic equations, this MONTHLY, vol. 61, 1954, pp. 340–342.
2. W. V. Lovitt, Elementary Theory of Equations. Prentice-Hall, New York, 1950.

PREDETERMINED DETERMINANT VALUES USING FIBONACCI NUMBERS

J. H. MEANS, Huston-Tillotson College

In plane analytic geometry the condition that three lines intersect in a common point is that the determinant

$$\begin{vmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{vmatrix} = 0.$$

If the instructor wishes to obtain a number of such lines he may do so by using the Fibonacci numbers. The well-known Fibonacci series is 1, 2, 3, 5, 8, $\cdots$, the law of formation being that any term after the second is the sum of the two preceding terms. Let the terms be denoted by U_1, U_2, U_3, $\cdots$ with U_0 being defined as unity, then $U_{n+2}=U_{n+1}+U_n$. Then we have

$$\begin{vmatrix} U_n & U_{n+1} & U_{n+2} \\ U_{n+1} & U_{n+2} & U_{n+3} \\ U_{n+2} & U_{n+3} & U_{n+4} \end{vmatrix} = 0, \qquad \text{for } n = 0, 1, 2, 3, \cdots$$

Example:

$$\begin{vmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 2 & 3 & 5 \end{vmatrix} = 0.$$

The three equations of the lines which intersect in a common point are

$$\begin{aligned} x + \ y + 2 &= 0 \\ x + 2y + 3 &= 0 \\ 2x + 3y + 5 &= 0. \end{aligned}$$

The instructor may use the following scheme to arrive at a similar result.

$$\begin{vmatrix} U_n & U_{n+2} & U_{n+4} \\ U_{n+1} & U_{n+3} & U_{n+5} \\ U_{n+2} & U_{n+4} & U_{n+6} \end{vmatrix} = 0.$$

Example:

$$\begin{vmatrix} 3 & 8 & 21 \\ 5 & 13 & 34 \\ 8 & 21 & 55 \end{vmatrix} = 0.$$

In plane analytic geometry where the area of a triangle is given by a third order determinant, a predetermined area may be handily obtained like this:

$$\begin{vmatrix} NU_n & U_{n+1} & 1 \\ NU_{n+1} & U_{n+2} & 1 \\ NU_{n+2} & U_{n+3} & 1 \end{vmatrix} = (-1)^{n+1}N, \qquad \text{where } N \text{ is any number.}$$

Example (with $N=4$):

$$\begin{vmatrix} 12 & 5 & 1 \\ 20 & 8 & 1 \\ 32 & 13 & 1 \end{vmatrix} = (-1)^4(4).$$

If the coordinates of the vertices of a triangle are (12, 5), (20, 8), and (32, 13) the area of the triangle is 2.

SOME COMMENTS ON A DIFFERENCE EQUATION

HARRY LEVY, University of Illinois

In a recent note in this MONTHLY,* Brand presents an elementary discussion of the distribution of cluster points of a difference equation of Riccati type. Since the problem possesses a simple geometrical interpretation, it would seem that the following discussion would be of interest.

The difference equation in question is given by

$$x_{n+1} = \frac{ax_n + b}{cx_n + d}, \qquad c \neq 0, \qquad D = \begin{vmatrix} a & b \\ c & d \end{vmatrix} \neq 0. \tag{1}$$

We shall drop the condition $c \neq 0$, and consider the projectivity τ

$$\tau\colon\ x' = \frac{ax + b}{cx + d}, \qquad D \neq 0, \tag{2}$$

as a point transformation in $S_1(R)$, a real projective space of one dimension, so that $x = \infty$ is an admissable value of x. Equation (1) then becomes

$$x_{n+1} = \tau x_n, \qquad x_0 = x, \tag{3}$$

so that

$$x_n = \tau^n x_0. \tag{4}$$

Thus the problem Brand considers can be rephrased: what can be said about the sequence (4) of successive transforms of a point x of $S_1(R)$ under an arbitrary projectivity?

It is quite easy to solve this problem in an elementary manner that presupposes no knowledge of projective geometry by considering the equation

$$cx^2 + (d - a)x - b = 0 \tag{5}$$

that determines the fixed points of τ. We can introduce a new projective coordinate y in terms of which the fixed points of τ are

$$(a)\ 0, \infty, \qquad (b)\ \infty, \infty, \qquad (c)\ \pm i \quad (i^2 = -1), \tag{6}$$

and thereby obtain the canonical forms of projectivities, corresponding to the three types of solutions of (5),

$$y' = ky, \qquad |k| \geqq 1, \tag{7a}$$

$$y' = y + 1, \tag{7b}$$

$$y' = \frac{\alpha y + \beta}{-\beta y + \alpha}. \tag{7c}$$

* Vol. 62, 1955, pp. 489–492.

Since (3) holds, we can express the sequence (1) in terms of the variable y. If τ is hyperbolic or parabolic we obtain from (7a, b) respectively, that

$$(8) \qquad \text{(a) } y_n = k^n y, \qquad \text{(b) } y_n = y + n.$$

If τ is elliptic, we regard it as an elliptic displacement and write (7c) in a more convenient form by letting

$$y = \tan\theta, \qquad y_n = \tan\theta_n, \qquad \alpha:\beta = \cos\omega:\sin\omega,$$
$$0 \leqq \theta < \pi, \qquad 0 \leqq \omega < \pi.$$

Then the projectivity τ becomes $\theta' = \theta + \omega \pmod{\pi}$ and θ_n satisfies

$$(8c) \qquad \theta_n = \theta + n\omega \pmod{\pi}.$$

The cluster points of the sequences (8a, b, c) are easily found from these equations. To state the results in terms of the sequence $\{x_n\}$, we first note that in the hyperbolic case the fixed points of (2) are k_1, k_2 given by

$$(9) \qquad k_1 = \frac{1}{2c}(a - d + R), \qquad k_2 = \frac{1}{2c}(a - d - R), \qquad R = \{(a-d)^2 + 4bc\}^{1/2}.$$

(If $c = 0$, we can understand by k_1, k_2 their limits as $c \to 0$.) The constant k of (7a) is the characteristic of τ, the cross ratio of the fixed points, a third point and its transform, so that*

$$(10) \qquad k = (k_1 k_2, xx') = \frac{a + d + R}{a + d - R}.$$

Thus the condition $|k| \geqq 1$ (which represents an ordering of the fixed elements k_1, k_2) is equivalent to a choice of a factor of proportionality in the coefficients of (2) for which

$$(11) \qquad a + d \geqq 0.$$

In the elliptic case, the invariant cross ratio is given by

$$(12) \qquad e^{2i\omega} = \frac{a + d + iR'}{a + d - iR'}, \qquad i^2 = -1, \qquad R' = \{-(a-d)^2 - 4bc\}^{1/2}.$$

We can then state the results as follows:

If (2) is a parabolic projectivity, the sequence (1) converges, independently of the initial value, to the unique solution of (5).

If (2) is a noninvolutory hyperbolic projectivity, the sequence $\{x_n\}$ converges to k_1 for all $x_0 \neq k_2$, where (9) and (11) hold. If $x_0 = k_2$, $x_n = k_2$ for all n.

If (2) is a hyperbolic involution, or if (2) is an elliptic projectivity and ω given by (12) is a rational multiple of π, then x_n is a periodic function of n,

* Graustein, Introduction to Higher Geometry, p. 149.

with period independent of the initial value x_0 (for $x_0 \neq k_1, k_2$).

If (2) is an elliptic projectivity with ω not a rational multiple of π, then $\{x_n\}$ (for a fixed real x_0) is everywhere dense.

Three further observations may be made. First, if in the sequence $\{x_n(x_0)\}$ determined by an initial value $x_0(\neq k_1, k_2)$, there exist some two terms that are equal

$$x_N(x_0) = x_{N+M}(x_0), \tag{13}$$

then $x_n(x_0)$ is a periodic function of n, for all x_0—for (13) implies that τ^M leaves fixed x_0 as well as the elements fixed under τ and hence τ^M is the identity. Second, if we consider the sequences $\{x_{-n}\}$ $(n>0)$, we note that the set of cluster points of $\{x_{-n}\}$ is of the same type as that of $\{x_n\}$, and that $\{x_{-n}\}$ converges if and only if $\{x_n\}$ converges. If τ is parabolic, both sequences converge to the same value, but if τ is hyperbolic and not an involution, $\lim x_{-n} = k_2$, where k_2 is the fixed element $\neq k_1$. Finally, we note that our geometric analysis of the sequence (1) can be made, without significant modification, when the variable and the parameters in (1) are complex. In complex projective space, a projectivity τ is either parabolic or nonparabolic. If τ is parabolic, it follows from (7a) that $\{x_n\}$ converges to the unique double element of τ, determined by (5). If τ is nonparabolic, we see from (7a) that we must distinguish between $|k| \neq 1$ and $|k| = 1$. If $|k| \neq 1$, we can suppose that $|k| > 1$, so that $y_n \to \infty$. Then $\{x_n\}$ converges to the solution k_1 of (5) given by (9) provided that the ambiguity in the determination of R in (9) is resolved by choosing for R the root that makes k, in (10), have absolute value greater than one. If, on the other hand, $|k| = 1$, we have two possibilities: either y_n is a periodic function of n or the values of y_n are everywhere dense on the circle $|y| = |y_0|$. But in the original x coordinate system, the circle $|y| = |y_0|$ is clearly the circle through x_0 orthogonal to the pencil of circles containing the two solutions of (5).

FINDING THE CARTESIAN EQUATION OF A LOCUS

E. B. Leach, Case Institute of Technology

Here is a method of finding an equation in rectangular coordinates of a plane locus for which we are given an equation in polar coordinates. It may be applied to a locus having a polar equation whose sides are polynomials in r and the sines and cosines of rational multiples of θ. The idea is to find an equation of the locus in r^2, $r\cos\theta$, and $r\sin\theta$, so that the Cartesian equation can be found simply by identifying these with x^2+y^2, x, and y, respectively.

Let $g(r, \theta) = 0$ be the given equation. If we replace θ by $\theta+\pi$ and r by $-r$, we get another equation of the same locus, $g_1(r, \theta) = 0$. On iterating the process we get a sequence of equations, each determining the same locus:

$$g(r, \theta) = 0, \quad g_1(r, \theta) = 0, \cdots, g_k(r, \theta) = 0,$$

k being chosen so that the transformation applied to $g_k(r, \theta)$ will give either $g(r, \theta)$, or $-g(r, \theta)$. We then form the product

$$G(r, \theta) = g(r, \theta) \cdot g_1(r, \theta) \cdot \cdots \cdot g_k(r, \theta).$$

The locus $G(r, \theta) = 0$ is evidently equivalent to the original given locus, and further, is not altered except for change of sign if we replace θ by $\theta + \pi$ and r by $-r$. It then appears that the coefficients of powers of r in $G(r, \theta)$ are periodic of period 2π, and so can be written in terms of $\sin \theta$ and $\cos \theta$. If G is then multiplied by a suitable power of r, the result can be expressed in terms of r^2, $r \cos \theta$ and $r \sin \theta$, and the identification described above can be made.

Example 1. $r = 1 + \cos \theta$. The two distinct forms given by the transformation are:

$$r - 1 - \cos \theta = 0$$

$$-r - 1 + \cos \theta = 0.$$

On multiplying these together and simplifying, we obtain:

$$\sin^2 \theta + 2r \cos \theta - r^2 = 0.$$

This is then multiplied by r^2 and we immediately see the Cartesian form of the equation:

$$y^2 + 2x(x^2 + y^2) - (x^2 + y^2)^2 = 0.$$

Example 2. $r = \sin(\theta/3)$. The different polar forms of the locus are:

$$r - \sin \frac{\theta}{3} = 0$$

$$-r - \sin \frac{\theta + \pi}{3} = 0$$

$$r - \sin \frac{\theta + 2\pi}{3} = 0.$$

Multiplying these together gives the complicated expression:

$$r^3 + r^2\left(-\sin \frac{\theta}{3} + \sin \frac{\theta + \pi}{3} - \sin \frac{\theta + 2\pi}{3}\right) + r\left(-\sin \frac{\theta + \pi}{3} \sin \frac{\theta + 2\pi}{3} + \sin \frac{\theta}{3} \sin \frac{\theta + 2\pi}{3} - \sin \frac{\theta}{3} \sin \frac{\theta + \pi}{3}\right) + \sin \frac{\theta}{3} \sin \frac{\theta + \pi}{3} \sin \frac{\theta + 2\pi}{3} = 0.$$

This can be simplified with the aid of various trigonometrical formulas to:

$$r^3 - \tfrac{3}{4}r + \tfrac{1}{4} \sin \theta = 0.$$

Then the Cartesian equation is found as in the first example to be:

$$(x^2 + y^2)^2 - \tfrac{3}{4}(x^2 + y^2) + \tfrac{1}{4}y = 0.$$

In the first example, it is seen that the main advantage of this method is that it avoids rationalization of an algebraic expression. The second example would be extremely difficult to carry out without the aid of a special device.

It should be noted that when multiplication by a power of r is required, the origin may be added to the original locus. In the above examples, the origin was already part of the locus, and so no alteration occurred. But the equation $r \sin\theta \cos^2\theta - 1 = 0$ gives a simple case where the origin appears in the Cartesian equation of the locus, but not in the polar equation.

THE EUCLIDEAN ALGORITHM

PHILIP FRANKLIN, Massachusetts Institute of Technology

Several elementary texts on number theory, such as the recent one by Harriet Griffin (1954) and the older one by R. D. Carmichael (1914), include the following as an exercise.

Prove that the number of divisions required to find the greatest common divisor of two positive integers written in the scale of 10 by means of the Euclidean algorithm does not exceed five times the number of digits in the smaller integer.

Both authors recognized this as a hard problem by its position at the end of a set, and Carmichael added a star. Since its difficulty for students is confirmed by experience in the classroom, a short solution may be of interest to teachers or students of number theory.

We first observe that a typical step of Euclid's algorithm at any stage uses the division transformation $a = qb + r$ to replace the pair a, b with $a > b$ by b, r with $b > r$. If $r < (5/8)b$, this single step reduces the smaller number in the ratio $r/b < 5/8$.

Next suppose that $r \geqq (5/8)b$. Then the following step of the algorithm uses $b = q_1 r + r_1$ with $q_1 = 1$ and $r_1 = b - r < (3/8)b$. This leads to the pair of numbers r, r_1 with $r > r_1$. And the smaller number has been reduced in the ratio $r_1/b \leqq 3/8$ for two steps. This is an average ratio of reduction less than 5/8 per step, since $3/8 < (5/8)^2$.

For five steps, the average ratio of reduction is less than $(5/8)^5 = 10^5/2^{10} = (1/10)(1000/1024)^2 < 1/10$. Hence for $5n$ steps the ratio will be less than 10^{-n}. Let n be the number of digits in the smaller integer of the original pair. Then the smaller integer initially had a value less than 10^n. Thus it would be reduced to less than unity if $5n$ steps were required. But the process stops when the remainder used as the new smaller number is zero. This proves that at most $5n$ steps will suffice.

The algorithm may be modified by using positive or negative remainders. In this case for a typical step we use $a = qb + r$ with $|r| \leqq b/2$ to replace the pair a, b with $a > b$ by $b, |r|$ with $b > |r|$. If $|r| < (3/7)b$, the first step reduces the smaller number in the ratio $|r|/b < 3/7$.

Next suppose that $|r| \geq (3/7)b$. Then the following step of the algorithm uses $b = q_1|r| + r_1$, with $q_1 = 2$. Since $|r| \leq b/2$, $r_1 \geq 0$. And $r_1 = b - 2|r| < b - (6/7)b = b/7$. This leads to the pair of numbers $|r|$, r_1. And the smaller number has been reduced in the ratio $r_1/b \leq 1/7$ for two steps. This is an average ratio of reduction less than $3/7$ per step, since $1/7 < (3/7)^2$.

For three steps, the average ratio of reduction is less than $(3/7)^3 = 27/343 < 1/10$. Hence for $3n$ steps the ratio will be less than 10^{-n}. This shows that if n is the number of digits in the smaller integer of the original pair, for the modified algorithm $3n$ steps will suffice.

The result for the Euclidean algorithm was found by G. Lamé in 1844. He used properties of the Fibonacci series in the proof. The result for the modified algorithm was found by Lionnet in 1857. Other references and details are given in Dickson's *History of the Theory of Numbers*, vol. I, pages 332 and 394.

ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY HOWARD EVES, University of Maine

Send all communications concerning Elementary Problems and Solutions to Howard Eves, Mathematics Department, University of Maine, Orono, Maine. This department welcomes problems believed to be new and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 1236. *Proposed by Hazel E. Evans, University of Pittsburgh*

For $a > b$ and $N < ab$ find the maximum value of N for which the equation

$$ax + by = N$$

has a solution in non-negative integers.

E 1237. *Proposed by Viktors Linis, University of Ottawa*

Let E be an ellipse, r_1 and r_2 focal radii, α the angle between the focal radii, and ds the element of arc. Evaluate the integrals

$$\int_E ds/(r_1 r_2)^{1/2} \quad \text{and} \quad \int_E (\cos \alpha/2)ds.$$

E 1238. *Proposed by M. S. Klamkin, Polytechnic Institute of Brooklyn*

Determine integral values of $n>0$ such that 3^n, 3^{n+1}, 3^{n+2} all have the same number of digits in their denary expansions.

E 1239. *Proposed by Josef Langr, Prague, Czechoslovakia*

Let $Q'\equiv A'B'C'D'$ be the quadrangle formed by the orthocenters A', B', C', D' of triangles BCD, CDA, DAB, ABC of a given convex quadrangle $Q\equiv ABCD$. Show that: (1) the vertices of Q and Q' lie on a common equilateral hyperbola, (2) Q and Q' have equal areas.

E 1240. *Proposed by H. Lindgren, Patent Office, Canberra, Australia*

Find six-piece dissections of a regular dodecagon into a square and into a Greek cross.

SOLUTIONS

Diophantine Consideration of the Optic Formula

E 1206 [1956, 186]. *Proposed by W. E. Briggs, University of Colorado*

If n is a given positive integer, how many integral solutions does

$$1/n = 1/x + 1/y$$

have with x and y positive and unequal?

Solution by N. J. Fine, University of Pennsylvania. Put $x=n+r$, $y=n+s$, with $r\neq s$. The given equation reduces to $n^2=rs$. Thus the required number of solutions is $d(n^2)-1$, where $d(N)$ is the number of divisors of N.

Also solved by J. L. Alperin, Leon Bankoff, G. E. Bardwell, H. W. Becker, D. M. Bloom, B. J. Boyer, A. L. Brown, D. H. Browne, Leonard Carlitz, W. B. Carver, George Cherlin, A. E. Danese, J. E. D'Atri and Stephen Lichtenbaum (jointly), G. W. Day, Monte Dernham, David DeVol, F. J. Duarte, Hazel E. Evans, Michael Goldberg, A. J. Goldman, D. S. Greenstein, Cornelius Groenewoud, E. H. Grossman, Nathaniel Grossmann, Virginia S. Hanly, B. A. Hausmann, R. E. Heaton, J. M. Howell, A. R. Hyde, F. I. John, M. S. Klamkin, Sam Kravitz, Aaron Lieberman, D. C. B. Marsh, Erich Michalup, J. B. Muskat, C. S. Ogilvy, Hyman Orlin, C. F. Pinzka, M. A. Rashid, L. A. Ringenberg, Azriel Rosenfeld, J. W. Ross, C. M. Sandwick, Sr., Erwin Schmid, J. P. Scholz, John Selfridge, G. J. Simmons, Malcolm Smith, Art Steger, G. B. Thomas, G. W. Walker, Chih-yi Wang, R. M. Warten, William Zlot, and the proposer.

Becker called attention to L. E. Dickson, *History of the Theory of Numbers*, vol. II, pp. 688–691. Thomas expressed interest in the related equation

$$1/n^2 = 1/x^2 + 1/y^2.$$

A Locus Problem

E 1207 [1956, 186]. *Proposed by R. D. Gordon, California State Polytechnic College*

Given two mutually perpendicular lines l_1 and l_2 in a plane, and a point Q in the plane located equally distant from l_1 and l_2. Determine the locus of a point P in the plane if its distance from Q equals the sum of its distances from l_1 and l_2.

Solution by N. G. Gunderson, University of Rochester. Choose axes and units so that l_1 is the y-axis, l_2 is the x-axis, and Q is at (1, 1). Then $P:(x, y)$ is on the locus if and only if

$$|x| + |y| = [(x-1)^2 + (y-1)^2]^{1/2},$$

or, squaring, if and only if $|xy| = -x-y+1$, and so, if and only if it is a point on the hyperbola $(x+1)(y+1)=2$ for which $xy \geqq 0$ or else a point on the lines $(x-1)(y-1)=0$ for which $xy \leqq 0$.

Also solved by A. N. Aheart, J. L. Alperin, W. A. Al-Salam, Robert Bart, Julian Braun, D. H. Browne, W. B. Carver, A. E. Danese, J. E. Darraugh, J. E. D'Atri, A. G. Davis, Hüseyin Demir, Hazel E. Evans, David Freedman, Herta T. Freitag, Michael Goldberg, A. J. Goldman, Cornelius Groenewoud, E. H. Grossman, J. H. Holdsworth, R. A. Hoodes, R. H. D. Hou, A. R. Hyde, I. M. Isaacs, F. I. John, Virginia L. Johnson, M. S. Klamkin, D. C. B. Marsh, Beckham Martin, Morris Morduchow, C. S. Ogilvy, Hyman Orlin, D. J. Peterson, M. A. Rashid, B. E. Rhoades, Azriel Rosenfeld, J. W. Ross, Erwin Schmid, Sister M. Stephanie, Malcolm Smith, Chih-yi Wang, G. W. Walker, David Zeitlin, and the proposer.

This problem illustrates the pitfalls of "squaring out," and for this reason has been given by the Proposer to his classes as a *prize problem*. Many of the above solutions were only partially correct.

Covering a Closed Interval

E 1208 [1956, 186]. *Proposed by Gretchen Geller, Student, University of California, Davis, Calif.*

Can a closed interval I be covered by an infinite set J of nondegenerate closed intervals in such a manner that no finite subset of J covers I?

Solution by J. L. Brown, Jr., Pennsylvania State University. Let I be the closed interval $[-1, 1]$. Then the infinite sequence of nondegenerate closed intervals

$$[-1, 0], [1/2, 1,], [1/4, 1], \cdots, [1/2^n, 1], \cdots$$

has the required property.

Also solved by Peter Crawley, A. E. Danese, N. J. Fine, A. J. Goldman, R. L. Helmbold, P. G. Hodge, Jr., F. A. Homann, L. E. Isenecker, F. I. John,

B. E. Rhoades, L. A. Ringenberg, J. W. Ross, Malcolm Smith, David Zeitlin, and the proposer. Late solution by M. S. Klamkin.

This problem occurs, without solution, as prob. *n*, p. 32, in M. E. Monroe, *Introduction to Measure and Integration.*

A Cross Ratio Associated with Any Triangle

E 1209 [1956, 186]. *Proposed by Hüseyin Demir, Zonguldak, Turkey*

Let ABC be any triangle and (I) its incircle. Let (I) touch BC, CA, AB at D, E, F, and intersect the cevians BE, CF at E', F' respectively. Show that the anharmonic ratio $D(E, F, E', F')$ is the same for all triangles ABC.

I. *Solution by W. B. Carver, Cornell University.* This is obviously a metrically special case of a more general projective theorem. The incircle may be replaced by *any conic* tangent to the sides at D, E, F, with the conic cutting the lines BE and CF at E' and F' respectively. By one of the limiting cases of Brianchon's theorem the lines AD, BE, CF meet in a point G. We set up a homogeneous coordinate system with A, B, C, G as the points (1, 0, 0), (0, 1, 0), (0, 0, 1), (1, 1, 1) respectively. It then follows readily that D, E, F are the points (0, 1, 1), (1, 0, 1), (1, 1, 0); the conic has the equation

$$x^2 + y^2 + z^2 - 2yz - 2zx - 2xy = 0;$$

E', F' are the points (1, 4, 1), (1, 1, 4); the lines through D have the equations $kx+y-z=0$ with $k=1, -1, -3, 3$ for DE, DF, DE', DF' respectively; and the required anharmonic ratio is therefore

$$(1 + 3)(-1 - 3)/(1 - 3)(-1 + 3) = 4.$$

II. *Solution by M. S. Klamkin, Polytechnic Institute of Brooklyn.* By a central projection, triangle ABC and its incircle (I) can be transformed into an equilateral triangle and its incircle. The anharmonic ratio $D(E, F, E', F')$ is invariant under this transformation and consequently is constant for all triangles. It is easy to show that $D(E, F, E', F')=4$.

Also solved by N. A. Court, P. W. M. John, D. C. B. Marsh, O. J. Ramler, Roscoe Woods, and the proposer.

A Dissection of a Pair of Equilateral Triangles

E 1210 [1956, 186]. *Proposed by Michael Goldberg, Washington, D. C.*

Given two equilateral triangles of edges a and b. Show how to dissect them by straight cuts into a total of no more than six pieces which can be assembled into another equilateral triangle. When the ratio of the larger to the smaller satisfies $a/b \geqq \sqrt{3}$, then five pieces suffice. When $a/b=4/3$, four pieces suffice.

Solution by H. Lindgren, Patent Office, Canberra, Australia. The maximum number of pieces required is not six but five.

When $a/b=4/3$, the triangle of side b is dissected as in Figure 1 into a trape-

zoid, and the other triangle is placed on AB. This is the special case for which four pieces suffice.

When $0<a/b<4/3$, the triangle of side b is dissected as in Figure 2 into a trapezoid whose top edge is equal to a, and again the other triangle is placed on AB. We have

$$AC = (a^2 + b^2)^{1/2} - a,$$

and the cuts are determined as follows:

$$DE = AC, \quad EF = FG, \quad GH = AC, \quad HK = KL.$$

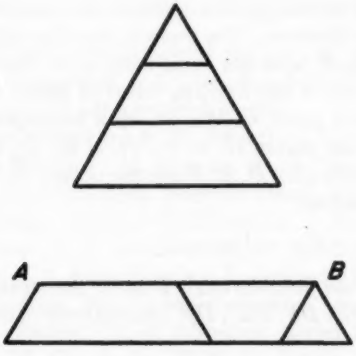

FIG. 1

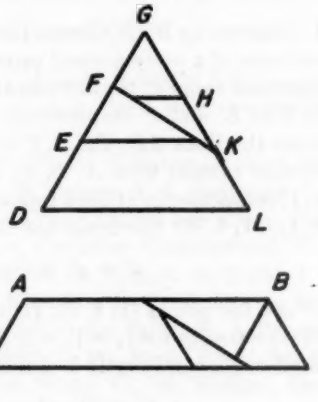

FIG. 2

This is the general case for which five pieces suffice. It is possible if F is not lower than H and K not higher than E. (The relative heights of E and H are immaterial.) Therefore it is possible if

$$DE \geqq (DG)/3,$$

that is, if

$$(a^2 + b^2)^{1/2} - a \geqq b/3,$$
$$a^2 + b^2 \geqq a^2 + 2ab/3 + b^2/9,$$
$$4b \geqq 3a.$$

The edges of the triangles have been named so that this condition is satisfied.

Also solved by C. F. Pinzka and the proposer.

By an alternative method, called *superposition of strips*, Lindgren obtained other dissections. In the dissections of Pinzka and the Proposer, only the larger triangle is cut. Pinzka's dissection requires only five pieces when $a/b>4/3$. W. B. Carver pointed out that four pieces are sufficient when $a/b=\sqrt{3}$; one has merely to bisect each triangle by a cut along an altitude.

ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY E. P. STARKE, Rutgers University

Send all communications concerning Advanced Problems and Solutions to E. P. Starke, Rutgers University, New Brunswick, New Jersey. All manuscripts should be typewritten with double spacing and margins at least one inch wide. Problems containing results believed to be new or extensions of old results are especially sought. Proposers of problems should also enclose any solutions or information that will assist the editor. In general, problems in well known textbooks or results in readily accessible sources should not be proposed for this department.

PROBLEMS FOR SOLUTION

4708. *Proposed by H. A. Bender, University of Rhode Island*

Prove the following relationship:

$$\int_0^1 e^{c(1-x^2)}dx = \sum_{n=0}^{\infty} \frac{2^n c^n}{3 \cdot 5 \cdot \cdots \cdot (2n+1)}.$$

4709. *Proposed by R. W. Marsh, Arlington, Va. and A. M. Gleason, Harvard University*

Let $f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_0$ be an irreducible polynomial with coefficients in the finite field of q elements. Suppose that the roots of f are primitive; that is, they generate the multiplicative group of their extension field. Prove that

$$F(x) = x^{q^n-1} + a_{n-1}x^{q^{n-1}-1} + \cdots + a_0$$

is irreducible.

4710. *Proposed by Hüseyin Demir, Zonguldak, Turkey*

Prove that if in a complete quadrangle inscribed in a circle (O) one pair of opposite sides are isotomic lines with respect to a triangle inscribed in (O), then the remaining pairs of opposite sides are also isotomic lines with respect to the same triangle.

4711. *Proposed by Morgan Ward, California Institute of Technology*

If $\pi(n)$ is the number of prime numbers not greater than n, show that

$$\pi(n) < \frac{5}{4} \frac{n}{\log n}$$

for every composite number n.

4712. *Proposed by J. V. Whittaker, University of California, Los Angeles*

Show that if $x_i \geq 0$ $(i = 1, \cdots, n)$ and $\sum_{i=1}^{n} 1/(1+x_i) \leq 1$, then $\sum_{i=1}^{n} 2^{-x_i} \leq 1$.

SOLUTIONS

Power Series Conditionally Convergent on the Unit Circle

4648 [1955, 496]. *Proposed by W. S. Loud, University of Minnesota*

A power series, $\sum_{n=0}^{\infty} a_n z^n$, has radius of convergence unity. It is known to converge at every point of its circle of convergence. Does it necessarily converge absolutely on the circle of convergence?

Editorial Note. It seems to be well known that a negative answer is called for. The following references are supplied by our contributors.

G. H. Hardy has shown that if we let $(1-z)^{-\frac{1}{2}} = \sum b_n z^n$, then the series $\sum (b_n/\log n) z^n$ converges conditionally and also uniformly on the circle of convergence. (See Dienes, *The Taylor Series*, pp. 464–465.)

Actually, problem 4449 [1952, 650] is equivalent to the present problem. There the series of Pringsheim is cited:

$$\sum_{n=1}^{\infty} (-1)^{[\sqrt{n}]} z^n/n.$$

It is convergent at every point of the unit circle. See also Pringsheim, *Vorlesungen über Funktionenlehre*, V. II, part 1, pp. 250–252; Knopp, *Theory and Application of Infinite Series*, p. 454; Knopp, *Problem Book in the Theory of Functions*, p. 31.

Another example is given by F. Herzog and G. Piranian, *Duke Mathematical Journal* (1949), p. 532. For schlicht Taylor series which converge uniformly but not absolutely on the unit circle, see P. Erdös, F. Herzog and G. Piranian, *Pacific Journal of Mathematics* (1951) pp. 75–82; also D. Gaier, *Math. Zeitschrift* (1953), pp. 349–350.

Power Series with Uniformly Bounded Partial Sums

4649 [1955, 496]. *Proposed by D. J. Newman, Advanced Development Division, AVCO, Stratford, Conn.*

Does the fact that a power series has uniformly bounded partial sums on the circle $|z| = 1$ necessarily imply that it converges everywhere on $|z| = 1$?

Editorial Note. The conclusion is not necessary. References sent in by readers include: Herzog and Piranian, *Duke Mathematical Journal*, 20 (1953) pp. 45–50; Erdös, Herzog and Piranian, *Math. Scand.*, 2 (1954), pp. 262–264.

Change of Sign in a Determinant

4650 [1955, 496]. *Proposed by J. L. Massera, Instituto de Matematica Y Estadistica, Montevideo, Uruguay*

Let $p_1, \cdots, p_k$ be any fixed indices, $0 \leqq p_1 < p_2 < \cdots < p_k < n$ and let $D(x_1, \cdots, x_k)$ be the nth order determinant whose elements a_{ij} are defined by: $a_{ij} = x_h$ if $i+j = p_h+1$, $h = 1, \cdots, k$; $a_{ij} = -x_h$ if $i+j = n+p_h+1$, $h = 1, \cdots, k$; $a_{ij} = 0$ otherwise. Show that D, considered as a function of the real variables

$x_1, \cdots, x_k$, may change its sign only in the neighborhood of those values which satisfy one of the relations $x_1 \pm x_2 \pm \cdots \pm x_k = 0$.

Solution by the Proposer. D vanishes if and only if

$$x_1 z^{r_1} + x_2 z^{r_2} + \cdots + x_k z^{r_k} = 0, \tag{1}$$

where z is an nth root of -1. This follows either directly from the factorization of D given by R. F. Scott, Note on a determinant theorem of Mr. Glaisher's, *Quarterly Journal of Mathematics*, 17 (1880) pp. 129–132, or by writing down the Euler resultant of the system formed by (1) (considered as an equation of degree $n-1$) and $z^n+1=0$ and then making suitable elementary transformations in this resultant which show that it is equal to $\pm D$.

Let $q_k = p_k - p_i$; the x_k being real, we will have

$$\begin{aligned} x_1 + x_2 z^{q_2} + \cdots + x_k z^{q_k} &= 0, \\ x_1 + x_2 \bar{z}^{q_2} + \cdots + x_k \bar{z}^{q_k} &= 0. \end{aligned} \tag{2}$$

Suppose the point $(x_1, \cdots, x_k)$ in k-dimensional real space satisfies (2) where z is an nth root of -1. If the rank of the matrix

$$\begin{pmatrix} 1 & z^{q_2} \cdots z^{q_k} \\ 1 & \bar{z}^{q_2} \cdots \bar{z}^{q_k} \end{pmatrix}$$

is 2, the set of points which satisfy (2) is a linear variety of dimension $k-2$; and since this variety does not separate the k-dimensional space, in the neighborhood of such points D cannot change its sign. If the rank is 1, then $z^{q_k} = \bar{z}^{q_k}$ which implies $z^{q_k} = \pm 1$ and one of the relations $x_1 \pm x_2 \pm \cdots \pm x_k = 0$ is satisfied.

Equilateral Triangles on a Parallelogram

4651 [1955, 496]. *Proposed by the late Joseph Rosenbaum*

On the sides of a parallelogram $A_1A_2A_3A_4$, equilateral triangles $A_iA_{i+1}B_i$ are constructed exteriorly. Then equilateral triangles $B_iB_{i+1}C_i$ are constructed interiorly to $B_1B_2B_3B_4$. Prove that $C_1C_2C_3C_4$ coincides with $A_1A_2A_3A_4$.

Generalize to the case where the original parallelogram is replaced by a polygon of n sides.

Solution by Joseph Langr, Prague, Czechoslovakia. Let the sides of the given parallelogram be $A_1A_2 = A_3A_4 = a$, $A_2A_3 = A_4A_1 = b$, and let angle $A_4A_1A_2 = \theta$. The three triangles $A_1B_1B_4$, $A_4A_3B_4$, $A_2B_1A_3$ are congruent, having $A_1B_1 = A_3A_4 = A_2B_1 = a$, $A_1B_4 = A_4B_4 = A_2A_3 = b$, and angles $B_4A_1B_1 = B_4A_4A_3 = B_1A_2A_3 = \theta + 120°$. Thus $B_1B_4 = B_4A_3 = B_1A_3$ and $A_3 \equiv C_4$. In the same manner we can show that A_1, A_2, A_4 coincide with C_2, C_3, C_1.

The situation is not essentially different if the triangles $A_iA_{i+1}B_i$ are constructed interiorly with respect to parallelogram $A_1A_2A_3A_4$, and the proof goes through unaltered. One must note, however, that it depends upon the shape of the original parallelogram whether triangles $B_iB_{i+1}C_i$ must be constructed ex-

teriorly or interiorly (with respect to $B_1B_2B_3B_4$) in order to guarantee the coincidence of A_i with C_{i+1}.

The same argument shows that, if we start with $A_1A_2 \cdots A_n$ and generate $C_1C_2 \cdots C_n$ by the above construction, each C_i is the fourth vertex of the parallelogram two of whose sides are adjacent sides of the given n-gon.

Also solved by Michael Goldberg, and R. L. Helmbold.

Editorial Note. The generalization intended by the proposer is the following. *Let $A_1A_2 \cdots A_n$ be a projection of a regular n-gon, and on the sides A_iA_{i+1} as bases construct exteriorly isosceles triangles $A_iA_{i+1}B_i$, where the vertex angle B_i is such that*

$$\sin \frac{B_i}{2} = \frac{\sqrt{2}}{2} \sin \frac{\pi}{n};$$

and on the sides B_iB_{i+1} as bases construct interiorly isosceles triangles $B_iB_{i+1}C_i$ similar to $A_iA_{i+1}B_i$; then when n is even the n-gons $C_1C_2 \cdots C_n$ and $A_1A_2 \cdots A_n$ are coincident, and when n is odd the two n-gons are congruent with corresponding sides parallel.

A Theorem on Measure

4652 [1955, 496]. *Proposed by Albert Wilansky, Lehigh University*

Let f be a Lebesgue integrable function on an interval. Let t_n be the measure of the set of points x such that $|f(x)| > n$. Prove that $\sum t_n < \infty$.

Solution by S. I. Goldberg, Wayne University, Detroit. This is an immediate consequence of the following well known result given as a problem in Titchmarsh, *The Theory of Functions*, 2nd ed., p. 343: *If $f(x)$ is a measurable function over an interval and u_n is the measure of the set of points x such that $n-1 \leq f(x) < n$, the necessary and sufficient condition that $f(x)$ be integrable is that $\sum_{n=-\infty}^{\infty} |n| u_n$ be convergent.*

To see this, note that $\sum t_n = \sum |n| u_n$.

Also solved by P. T. Bateman, Joshua Barlaz, G. U. Brauer and Chih-yi Wang, W. M. Faucett, Harley Flanders, A. J. Goldman, C. D. Gorman, D. S. Greenstein, P. R. Halmos, J. Horváth, B. C. Kenny, William Kruskal, A. E. Livingston, Neill McShane, C. M. Pearcy, H. G. Tucker, and the Proposer.

Editorial Note. The result is given as an exercise (accompanied by a strong hint of the solution) in Halmos, *Measure Theory* (New York, 1950), p. 115. The exercise points out that the condition of problem 4652 is sufficient as well as necessary, and also, for complete precision, requires the interval to be bounded.

RECENT PUBLICATIONS

EDITED BY E. P. VANCE, Oberlin College

All new material for this section should be sent to the incoming editor, Richard V. Andree, University of Oklahoma, Norman, Oklahoma.

Fundamental Concepts of Geometry. By B. E. Meserve. Addison-Wesley Publishing Company, Cambridge, Massachusetts, 1955. ix+321 pages. $7.50.

In this work one finds material which should be valuable for anyone teaching mathematics, especially geometry, as it shows the relationship of several types of geometries to one another.

The aims of this book, according to the author, are to help the reader

1. Discover how euclidean plane geometry is related to, and often a special case of, many geometries,
2. Obtain a practical understanding of "proof,"
3. Obtain the concept of a geometry as a logical system based upon postulates and undefined elements, and,
4. Appreciate the historical evolution of our geometrical concepts and the relation of euclidean geometry to the space in which we live.

The first chapter of this text is devoted to a study of various sets of postulates, some of their consequences, and the associated logic. Then in Chapter Two synthetic projective geometry is developed from the postulational viewpoint. Later coordinate systems are introduced, and some of the basic properties of analytic projective geometry are developed. The relation of this geometry to the synthetic projective geometry already considered is then discussed. Affine geometry is next considered as a special case of analytic projective geometry, and euclidean geometry is developed as a special case of affine geometry. The last three chapters of the book are devoted to a historical sketch of the development of geometry, a brief discussion of some non-euclidean geometries, and a descriptive approach to topology.

Klein's definition of a geometry, "The study of the invariants of a configuration under a group of transformations" is frequently mentioned, and the emphasis is on the study of the transformations used in the various geometries and the invariants that exist under these transformations. Many of the basic properties of the various geometries are developed, but no attempt is made to study completely any one of the geometries under consideration. However, enough of each geometry is developed so that the restrictions which need to be imposed on it to obtain another geometry are evident.

From the pedagogical standpoint, it should be noted that the proofs of some of the theorems needed in the development of the text are left as exercises for

the student. Also, there is a paucity of illustrative examples, which might make it difficult for an embryonic mathematician to see what is wanted or how to proceed in some of the exercises. As far as mathematical techniques are concerned, only those of College Algebra are needed, but a greater mathematical background is needed for an appreciation of the material presented.

R. G. SANGER
Kansas State College

Analytic Geometry and Calculus. By T. S. Peterson. New York, Harpers Mathematics Series. Harper and Brothers, Publishers, 1955. 456 pages. $6.50.

Over-all picture of text. This text covers the analytic-geometry topics, the calculus topics, and the introduction-to-differential-equations found in traditional, approved analytic-geometry and calculus texts. Formulas and curves of more elementary mathematics together with tables, mathematical symbols, and the Greek alphabet are conveniently located. The order of topic-presentation and the pattern of exposition (theory, then illustrations, examples and solutions) are conventional.

Concerning the author's objectives. As to the design, purpose, and intended rigor, Peterson writes, "This book is designed to serve as an introductory course in Analytic Geometry and Calculus for students who have some familiarity with algebra and trigonometry," and "It is the purpose of the book to present Analytic-Geometry and Calculus not only as a powerful tool that can be used in applied fields but also as an important branch of mathematical analysis. To serve this latter purpose, all definitions, theorems, and general procedures are as accurately presented as the demands of clarity and simplicity permit."

Peterson achieves the objective of the first quotation by writing in a simple and concise manner. In fact Peterson writes so concisely that one is likely on first reading to conclude that the book is too abridged. However, on a second reading one realizes that there is still much material in this text. In fact, any student who thoroughly masters all the material in Peterson's text will be definitely superior to the average run-of-the-mill calculus student.

In the second quotation, the decision to develop rigor compatible with clarity and simplicity seems wise in view of the deficiency in self-discipline prevalent in today's students in many classes.

Concerning controversial definitions. Now by way of discussing a few topics that are of interest to writers and teachers, I present this paragraph.

In defining *function*, some authors emphasize that a function is the *relationship, law or correspondence between two variables*, or is a *set of number-pairs*. Peterson writes as follows, "If two variables x and y are related so that, for each x in a range R of real numbers, we obtain one or more real values for y, then y is said to be a real function of the real variable x defined over the range R." Possibly the presentation of limits on page 73 could be improved by elaboration. Peterson does not emphasize a distinction between integration and antidifferentiation. He does not distinguish between iterated and multiple integrals. In

writing today there are new ways of defining *versus* traditional ways of defining. Both can be criticized. Peterson in most cases adheres to the traditional.

Comments and conclusions. More youths are kept in school today, and hence more non-students are found in our classes. Also the multitudinous distractions in the form of extra-curricular activities have caused some of our good students to expect more of the text and the professor. Hence great moderation must be exerted in the extent of rigor, ramifications, and details when writing some texts. The reviewer feels that Peterson's text is very appropriate for many of today's students.

E. M. PEASE
University of Rhode Island

Analytic Geometry, By N. H. McCoy, and R. E. Johnson. New York, Rinehart and Company, 1955. 301+xiv pages. $4.00.

Analytic Geometry, New Second Edition, By R. R. Middlemiss. New York, McGraw-Hill Book Company, 1955. 310+ix pages. $3.75.

Each instructor has his own idea concerning the teaching of analytic geometry, and from the great wealth of textbooks, he is sure to find one to suit his purposes. As most students study analytic geometry as preparation for the calculus, the instructor is guided in his selection as to the purposes of his students, the basic ideas, and the presentation of the methods for graphing. A textbook then stands or falls according to the preferences of the instructor.

McCoy and Johnson in the preface state: "The emphasis throughout the book is on an understanding of the basic principles of analytic geometry." The text contains the traditional material, but in an order which does not always force the best understanding of the basic principles. In the discussion of conic sections, only the simplest standard forms are discussed very fully. Two chapters later, the standard form is again introduced after the translation of axes has been discussed. There seems to be a tendency throughout the textbook to encourage the students to substitute in memorized relations derived from proven theorems. For example, the center and the radius of the circle $x^2+y^2+Dx+Ey+F=0$ are found from the relations $(-D/2, -E/2)$ and $\frac{1}{2}\sqrt{D^2+E^2-4F}$ rather than the completion of the squares in each individual case.

The discussion of the general properties of curves in Chapter 3, entitled "Equations and Graphs", concerns itself with the intercepts, extent, and symmetry, but only of the conic sections. It is felt that the rapid sketching of curves is a most important contribution of analytic geometry to the student's future work in mathematics and should not be confined to only the simplest of curves. The authors seem however to give only "lip service" to these properties, since throughout the text extensive tables of values are compiled (some entries to three decimal places) and used in the sketching of algebraic and transcendental curves. These tables of values have a place in introducing rectangular coordinates in algebra courses but are not desirable in analytic geometry.

The format of the text is excellent in general, with large, clear type and excellent illustrations especially in the section devoted to solid analytic geometry. Some of the earlier illustrations, *i.e.*, in the chapter on polar coordinates, are too extensive and could gain from judicious elimination of excess information.

The preface of the textbook by Middlemiss sets forth his plan and arrangement, and states in part: "The original edition of this text was written with the primary aim of giving students a better preparation for calculus and the sciences. To this end, less than the usual amount of space was devoted to the conic sections—and more than the average amount was devoted to the polynomials and rational fractional functions and to the graphs and properties of the exponential, logarithmic, and trigonometric functions. . . . All these characteristics of the original text have been retained in this new edition."

A brief introduction compiled of essential definitions and formulas from algebra and trigonometry include the evaluation of determinants by minors, the definitions of trigonometric functions, and the fundamental relations between the trigonometric functions.

In Chapter 4, the way is paved easily to the general discussion of curves through the polynomials, while the more formalized discussion of intercepts, symmetry, extent, and asymptotes, is introduced in Chapter 5 using rational fractional functions. The commendable practice of orderly discussion of curves in the numerous examples throughout the text should greatly aid those students who have hitherto had only a vague notion of what was required in rapid sketching.

The introduction of transformation of coordinates before the discussion of the circle and the other conic sections permits a fuller discussion of the general standard forms together with the simpler forms.

A fine feature of the text is a more complete discussion of trigonometric, exponential, and logarithmic curves than is ordinarily found in analytic geometry texts. The essential properties of lines, planes, and surfaces required in the calculus have been included in the section on solid analytic geometry.

Students using the text by Middlemiss will have a clearer understanding of the nature of graphs and equations and a more solid foundation for future application and study. The less important aspects of the curves have been relegated to the problem sets, which are numerous and varied.

The format of the text is excellent, with clear-cut and ample illustrations.

E. F. MYERS
University of Pittsburgh

NEWS AND NOTICES

EDITED BY EDITH R. SCHNECKENBURGER, University of Buffalo

Readers are invited to contribute to the general interest of this department by sending news items to Edith R. Schneckenburger, University of Buffalo, Buffalo 14, New York. Items should be submitted at least two months before publication can take place.

MATHEMATICS DIVISION OF ASEE

The Mathematics Division of the American Society for Engineering Education met on June 25–29, 1956, at Iowa State College. Four well-attended sessions were held, including a joint conference with the Iowa Section of the Mathematical Association of America. The following new officers of the Division were elected at the annual business meeting: Chairman, Professor W. G. Warnock, Rensselaer Polytechnic Institute; Secretary, Professor W. E. Restemeyer, University of Cincinnati; Director, Professor G. B. Thomas, Massachusetts Institute of Technology; Director, Professor C. O. Oakley, Haverford College; Director and Representative to General Council of ASEE, Dr. R. S. Burington, Department of Navy. The next annual meeting of the Mathematics Division of the ASEE will be held in June, 1957, at Cornell University. For further information write to Professor W. E. Restemeyer, University of Cincinnati.

MATHEMATICS TEACHING ESSAY CONTEST

Kappa Mu Epsilon and the Science Teaching Improvement Program of the American Association for the Advancement of Science are cooperating in the sponsorship of an essay contest on "Opportunities in Teaching Mathematics in Secondary Schools." Satisfactory essays will be published in *The Pentagon*. First prize in the contest will be $50. There will be second and third prizes of $25 and $15, respectively.

The Mathematics Teaching Essay Contest is planned to increase interest in the teaching of mathematics at the secondary-school level, and to encourage all students in mathematics to consider the advantages of a career in secondary-school mathematics teaching. It is also hoped that the preparation, as well as the reading, of the essays may attract good students with an aptitude for and interest in mathematics to enter the teaching profession. The importance of the ability to express oneself in writing, particularly on the part of teachers, should also be emphasized by such an essay contest.

Essays submitted in the contest should reach Professor Carl V. Fronabarger, Southwest Missouri State College, Springfield, Missouri, no later than April 1, 1957. They must be not more than 1,000 words in length, and should be typed double-spaced on a good grade of paper. Four copies should be submitted by each contestant. The content of the essay should be as specific as possible, and point out the advantages of preparation for the teaching of mathematics at the secondary-school level. The essay may consider one or more of the special facets

of the profession of mathematics teaching, or cover the general area as completely as the length of the essay will permit. Undergraduate and graduate students in mathematics are eligible to enter the contest. The essays will be judged on accuracy and objectivity of the data presented, the degree to which the essay appears to be convincing in the case presented for mathematics teaching, and composition and neatness.

PRELIMINARY ACTUARIAL EXAMINATIONS PRIZE AWARDS

The winners of the prize awards offered by the Society of Actuaries to the nine undergraduates ranking highest on the score of Part 2 of the 1956 Preliminary Actuarial Examination are as follows:

First Prize of $200

Pratt, Richard L.	Washington University

Additional Prizes of $100 each

Brillinger, David R.	University of Toronto
Earle, Clifford J., Jr.	Swarthmore College
Kaplan, Stanley	Cornell University
Mosher, Robert E.	Kenyon College
Riehm, Carl R.	University of Toronto
Rubin, Jerrold	Columbia University
Schweitzer, Paul A.	Holy Cross College
Soderquist, George D.	Drake University

The Society of Actuaries has authorized a similar set of nine prizes for the 1957 examinations on Part 2.

The Preliminary Actuarial Examinations consist of the following three examinations:

Part 1. Language Aptitude Examination.
(Reading comprehension, meaning of words and word relationships, antonyms, and verbal reasoning.)

Part 2. General Mathematics Examination.
(Algebra, trigonometry, coordinate geometry, differential and integral calculus.)

Part 3. Special Mathematics Examination.
(Finite differences, probability and statistics.)

The 1957 Preliminary Actuarial Examinations will be prepared by the Educational Testing Service under the direction of a committee of actuaries and mathematicians and will be administered by the Society of Actuaries at centers throughout the United States and Canada on May 15, 1957. Applications should be submitted before April 1, 1957, to the Society of Actuaries, 208 South LaSalle Street, Chicago 4, Illinois.

PERSONAL ITEMS

Fresno State College announces the following: Dr. S. J. Bryant, formerly a teaching assistant at the University of Missouri, Dr. V. E. Howes, and Dr. T. C. Kipps of the University of Santa Clara have been appointed to instructorships; Assistant Professor G. D. Alkire has been promoted to an associate professorship.

Purdue University reports: Associate Professor R. A. Oesterle of Eastern Oregon College of Education has been appointed Assistant Professor of Mathematics and Education; Mr. G. D. Miller, previously principal of Shadeland High School, Indiana, and Dr. Judah Rosenblatt have been appointed to instructorships; Dr. Rosenblatt is also a consultant in the Statistical Laboratory.

At San Diego State College: Dr. C. V. Holmes of Northrop Aircraft and Assistant Professor Margaret Willerding of Harris Teachers College have been appointed to assistant professorships; Mr. Roger Shaw of the University of California has been appointed to an instructorship; Associate Professor A. R. Harvey has been promoted to a professorship; Dr. L. J. Warren has been promoted to an assistant professorship.

Southwest Texas State Teachers College announces: Mr. W. C. Akin, Mrs. Saphrona B. Krause, and Mr. A. W. Spear have been appointed to instructorships.

Mr. D. S. Adorno of the University of Utah has been appointed to an instructorship at Iowa State College.

Mr. F. T. Baker, previously a student at Yale University, has a position as a programmer with the International Business Machines Corporation, New York, New York.

Mr. J. M. Behr, formerly a technical writer for Librascope, Glendale, California, has a position as a computer applications engineer with the Royal Precision Corporation.

Assistant Professor Dean Clifton Benson of South Dakota School of Mines and Technology has been appointed to an assistant professorship at Chico State College.

Mr. W. S. Bicknell of Wayne University has been appointed Assistant Professor of Commerce and Actuarial Science at the University of Wisconsin.

Mr. I. W. Boxer of California State Polytechnic College has been appointed to an instructorship at El Camino College.

Mrs. Carolina M. Brennan has been appointed to an instructorship at the University of the Philippines.

Professor A. R. Brown, Jr. of Drury College has accepted a position as chief, Office of Ballistic Computations, Air Force Armament Center, Eglin Air Force Base, Florida.

Mr. Donald Brown, previously a student at Hofstra College, is employed now as an aeronautic research intern by the National Advisory Committee on Aeronautics, Langley Field, Virginia.

Mrs. Mary H. Brown of Florida State University has been appointed to an instructorship at Perkinston Junior College.

Assistant Professor W. O. Buschman of Portland State College has been appointed to an assistant professorship at California State Polytechnic College.

Mr. C. N. Campopiano, formerly a project engineer for the Sperry Gyroscope Company, Great Neck, New York, has accepted a position as a research associate at the Micro-wave Research Institute of the Polytechnic Institute of Brooklyn.

Associate Professor F. L. Celauro of East Tennessee College has been promoted to a professorship.

Mr. E. W. Cheney, recently a teaching fellow at the University of Kansas, has a position as a senior research engineer with Consolidated-Vultee Aircraft Corporation, San Diego, California.

Assistant Professor Eckford Cohen of the University of South Carolina has been appointed to an assistant professorship at the University of Tennessee.

Assistant Professor R. M. Conkling of the University of New Hampshire has been appointed to an assistant professorship at New Mexico College of Agriculture and Mechanic Arts.

Mr. B. G. Cooper, formerly a graduate assistant at Kansas State College, is a teacher at Coronado High School, California.

Dr. F. W. Donaldson, previously a senior aerophysics engineer for Consolidated-Vultee Aircraft Corporation, Fort Worth, Texas, is now Supervisor of the Computing Laboratory, General Electric Company, Philadelphia, Pennsylvania.

Mr. Willard Draisin, formerly a student at Brooklyn College, is employed as a staff member of Lincoln Laboratories, Lexington, Massachusetts.

Dr. Jacqueline P. Evans of Smith College has been appointed to an assistant professorship at Wellesley College.

Professor S. H. Gould of Williams College has been appointed Executive Editor of Mathematical Reviews.

Mr. E. P. Graney is employed as an analyst by the Westinghouse Electric Corporation, Kansas City, Missouri.

Dr. R. C. Gunning of the University of Chicago has been appointed a Higgins Lecturer at Princeton University.

Dr. N. A. Hall, assistant dean of the College of Engineering, New York University, has been appointed Professor and Chairman of the Department of Mechanical Engineering of Yale University.

Dr. Melvin Hausner of Brooklyn College has been appointed to an assistant professorship at Stevens Institute of Technology.

Mr. F. E. Hazlett, previously a computer applications engineer at Librascope, Glendale, California, has a position as a computer applications engineer with the Royal Precision Corporation.

Mr. Simon Hellerstein, formerly a graduate assistant at Syracuse University, has been appointed to an instructorship at the University.

Brigadier General Harris Jones, professor of mathematics and dean of the U. S. Military Academy, has retired.

Mr. Edward Keegan, previously a graduate student at Kansas State Teachers College, is employed as a mathematician by the Coast and Geodetic Survey, Washington, D. C.

Associate Professor L. G. Kelly of Clemson Agricultural College has been appointed a research scientist at Lockheed Missiles Systems Division, Mathematical and Computing Center, Palo Alto, California.

Dr. A. V. Kozak, chairman of the Department of Mathematics and engineering officer of Concord College, has been granted a leave of absence to do additional graduate study and to work as a member of the operations research staff at the Glenn L. Martin Company, Baltimore, Maryland.

Assistant Professor L. T. LaBorde of the University of the South has been appointed to an assistant professorship at the University of Cincinnati.

Captain W. T. Lee, recently a teacher at Oklahoma Military Academy, has a position as a nuclear engineer with the Glenn L. Martin Company, Baltimore, Maryland.

Mr. J. L. Lewis, Jr., has a position as an engineer for Westinghouse Electric Corporation, Baltimore, Maryland.

Mr. B. W. Marks, formerly a teacher at San Jacinto Junior High School, Midland, Texas, has a position as an analytical chemist at Continental Oil Company, Westlake, Louisiana.

Dr. Ceslovas Masaitis of the University of Tennessee has a position as a mathematician at the Ballistic Research Laboratory, Aberdeen Proving Ground, Maryland.

Dr. L. F. McAuley of the University of Maryland has been appointed to an instructorship at the University of Wisconsin.

Professor C. N. Mills of Augustana College has been appointed Visiting Lecturer at Florida State University.

Dr. C. V. Newsom, formerly Executive Vice-President, has been elected President of New York University. Dr. Newsom served as Editor of this MONTHLY from 1947 to 1951.

Dr. R. H. Owens, recently a mathematician at the Office of Naval Research, Washington, D. C., has been appointed to an assistant professorship at the University of New Hampshire.

Dr. Edgar Reich of the Rand Corporation has been appointed to an assistant professorship at the University of Minnesota.

Mr. L. A. Rife of Mississippi Southern College has been appointed to an assistant professorship at Clemson Agricultural College.

Dr. P. D. Ritger of Stevens Institute of Technology has been promoted to an assistant professorship.

Mr. D. H. Schmieder, formerly a graduate assistant at the University of Kentucky, is a physicist at the Army Ballistic Missile Agency, Huntsville, Alabama.

Miss Janet H. Scott (Mrs. Allsbrook), previously a graduate assistant at the University of South Carolina, is now a mathematician at the David Taylor Model Basin, Washington, D. C.

Mr. R. M. Shank, formerly a graduate student at the University of Kentucky, is teaching at Nitro High School, West Virginia.

Mr. G. J. Simmons, previously a graduate student at the University of Oklahoma, is employed as a research scientist at the Lockheed Missile Division, Van Nuys, California.

Mr. R. L. Slater, Jr., formerly an assistant engineer for Chicago Midway Laboratory, Chicago, Illinois, is now an associate laboratory director at Bjorksten Research Laboratories, Madison, Wisconsin.

Dr. Paul Slepian of Brown University is employed now as a member of the technical staff of the Ramo-Wooldridge Corporation, Los Angeles, California.

Assistant Professor R. W. Sloan of the University of New Hampshire has been appointed to an assistant professorship at Carleton College.

Mr. R. A. Spong, previously a mathematician at the U. S. Navy Underwater Sound Laboratory, New London, Connecticut, has accepted a position as a scientist-mathematician in the Special Projects Division of the Research and Development Department of the Electric Boat Division, General Dynamics Corporation, Groton, Connecticut.

Mr. Vivian Spurgeon has been appointed to an assistant professorship at East Texas Baptist College.

Dr. R. D. Stalley of Fresno State College has a position as a mathematician at the U. S. Naval Ordnance Test Station, China Lake, California.

Mr. Robert Tates, recently a graduate student at Syracuse University, is now an engineer with the Glenn L. Martin Company, Baltimore, Maryland.

Associate Professor H. E. Taylor of Florida State University is on leave during 1956–57 as a Carnegie Intern in General Education and is Visiting Associate Professor at the University of Chicago.

Mr. Jack Warga of the Electro Data Corporation, Pasadena, California, is on leave of absence on a Chaim Weizmann Memorial Fellowship at the Weizmann Institute of Science, Rehovoth, Israel.

Dr. Ti Yen of Lehigh University has been appointed to an assistant professorship at Illinois Institute of Technology.

President Emeritus F. C. Ferry of Hamilton College died on August 14, 1956. He was a charter member of the Association.

Professor Emeritus L. C. Karpinski of the University of Michigan died on January 25, 1956. He was a charter member of the Association.

Professor Emeritus William Marshall, who was formerly Head of the Department of Mathematics of Purdue University, died on July 31, 1956.

Sir Edmund T. Whittaker of Edinburgh, Scotland, died on March 24, 1956.

THE MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

THE THIRTY-SEVENTH SUMMER MEETING OF THE ASSOCIATION

The thirty-seventh summer meeting of the Mathematical Association of America was held at the University of Washington, Seattle, Washington, on Monday and Tuesday, August 20 and 21, 1956, in conjunction with the summer meetings of the American Mathematical Society, the Biometric Society, the Econometric Society, and the Institute of Mathematical Statistics. There were 919 persons registered, including 303 members of the Association.

Sessions of the Association were held on Monday morning and afternoon and on Tuesday morning in the Guggenheim Auditorium of the University of Washington. President W. L. Duren and former Vice-President F. L. Griffin presided on Monday morning, Professor W. A. Golomski on Monday afternoon, and Vice-President G. B. Price and President Duren on Tuesday morning. The fifth series of Earle Raymond Hedrick Lectures was delivered by Professor J. C. Oxtoby of Bryn Mawr College. The Program Committee for the meeting consisted of C. R. Wylie, Jr., Chairman, C. L. Clark, W. A. Golomski, and F. E. Hohn.

FIRST SESSION OF THE ASSOCIATION

The Earle Raymond Hedrick Lectures: "Category and Measure"; Lecture I, by Professor J. C. Oxtoby, Bryn Mawr College.

"The Metropolitan New York Mathematical Competition," by Professor W. H. Fagerstrom, City College of New York.

"The Stanford University Competitive Examination in Mathematics," by Professor H. M. Bacon, Stanford University.

"The William Lowell Putnam Mathematical Competition," by Professor L. E. Bush, Kent State University.

SECOND SESSION OF THE ASSOCIATION

Hedrick Lecture II, by Professor Oxtoby.

"Dynamic Programming," by Dr. Richard Bellman, Rand Corporation. (By title.)

"The Practice of Mathematics," by Dr. R. E. Gaskell, Boeing Airplane Co.

"Games of Survival," by Dr. L. S. Shapley, California Institute of Technology.

THIRD SESSION OF THE ASSOCIATION

Hedrick Lecture III, by Professor Oxtoby.

Business Meeting of the Association.

"Report from the Commission on Mathematics of the College Entrance Examination Board," by Dean A. E. Meder, Rutgers University, and Professor G. B. Thomas, Massachusetts Institute of Technology.

MEETING OF THE BOARD OF GOVERNORS

The Board of Governors of the Association met on Monday evening in the lounge of McKee Hall in the Women's Residence Hall, with twenty members present. Among the more important items of business transacted were the following:

The Board voted to accept the resignation of Professor Ralph Hull as Governor from the Indiana Section and elected Professor P. D. Edwards of Ball State Teachers College to the Board for the balance of the unexpired term.

The Secretary-Treasurer was instructed to prepare amendments to the By-Laws to be voted on at the annual meeting of December 1956 providing for a more specific statement of the control of the Association over its sections, and for an increase in the annual dues to five dollars and in the dues of emeritus members who receive the MONTHLY to two dollars. The Board also voted to increase the annual subscription rate of the MONTHLY to $6 effective January 1, 1957.

It was voted to hold the Forty-first Annual Meeting at the University of Cincinnati and the Hotel Sheraton-Gibson, Cincinnati, Ohio, on Friday, January 31, 1958.

Professor Leo Zippin of Queens College was invited to deliver the Hedrick Lectures at the 1957 Summer Meeting.

Approval was voted for the holding of a meeting for the organization of a New Jersey Section of the Association, at which tentative by-laws are to be adopted and officers elected.

The Board voted that the Association will sponsor an annual contest in mathematics for secondary school students in the United States and Canada, the first such contest to be held in 1958 under the supervision of a standing committee on High School Contests.

BUSINESS MEETING OF THE ASSOCIATION

At a business meeting held on Tuesday morning, announcement was made that a grant of $1,250 had been received from the National Science Foundation for the support of the work of the Committee on Mathematical Personnel and Education. It was also announced that the membership of the Association was 6,461 on August 6.

The following committee chairmen reported on the work of their respective committees: B. W. Jones for the Committee on Visiting Lecturers; W. L. Duren

for the Committee on the Undergraduate Program in Mathematics; G. B. Price for the Committee to Study the Activities of the Association; Tomlinson Fort for the Committee on Mathematical Personnel and Education; R. P. Bailey on High School Contests; and J. S. Frame for the Committee on Employment Opportunities.

MEETING OF SECTION OFFICERS

A meeting of officers of the Sections of the Association was held on Tuesday evening in the lounge of McKee Hall. About thirty persons were present representing 22 of the 26 sections.

Among the topics discussed were the various types of contests now being conducted by the sections, the recruitment of college mathematics teachers, revision of the undergraduate mathematics curriculum, and programs of sections. Emphasis was placed on traveling lectureships and other contacts with high school teachers and students in order to make known the many professional opportunities now open to mathematicians.

MEETINGS OF OTHER ORGANIZATIONS

The American Mathematical Society held its sessions from Tuesday afternoon through Friday afternoon. The colloquium lectures on "Harmonic analysis and probability" were delivered by Professor Salomon Bochner of Princeton University.

Meetings of the Biometric Society, the Econometric Society, and the Institute of Mathematical Statistics were also held during the week of August 20.

ARRANGEMENTS, ENTERTAINMENT, AND RECREATION

The Committee on Arrangements for the meeting consisted of: A. E. Livingston, Chairman; R. A. Beaumont, Z. W. Birnbaum, D. G. Chapman, D. B. Dekker, H. M. Gehman, V. L. Klee.

Registration headquarters was in the Student Union Building of the University of Washington. The Employment Register and the Book Exhibit were on display in the Union Building. Dormitory accommodations were available in the Women's Residence Hall and meals were served in the Union.

The facilities of the Officers' Club at the Sand Point Naval Air Station were available for an informal party on Tuesday evening. A salmon bake was held on Wednesday afternoon and evening at Seward Park. A boat cruise along the Seattle waterfront and around Bainbridge Island was held on Thursday.

A motion prepared by Professor Tomlinson Fort was adopted by the five mathematical organizations, expressing thanks to the members of the Department of Mathematics of the University of Washington and to the ladies of the department for their labor with hand and heart for the comfort and happiness of the visiting mathematicians.

H. M. GEHMAN, *Secretary-Treasurer*

NEW MEMBERS

Professor H. M. Gehman, Secretary-Treasurer, announces that the following 44 persons have been elected to membership by the Board of Governors on applications duly certified.

R. L. ADLER, A.M. (Columbia) United States Army.

E. A. ANDERSON, JR., Student, Lebanon Valley College.

SEYMOUR BACHMUTH, B.S. (Brooklyn) Brooklyn, New York.

D. J. BOEDICKER, B.S. (LeMoyne) Instrument Engr., International Business Machines Corp., Poughkeepsie, N. Y.

B. R. BUZBY, B.S. (Antioch) Grad. Student, Indiana University.

A. A. CAPORASO, M.S. (N.Y.U.) Math., International Business Machines Corp., New York, N. Y.

JULIANNE COLLINS, Student, Cardinal Stritch College.

C. M. CONDIT, Student, Idaho State College.

W. M. CUNNEA, S.M. (Chicago) Teaching Asst., University of Chicago.

F. C. DESUA, Ph.D. (Pittsburgh) Asst. Professor, University of Pittsburgh.

A. J. GRUBER, Student, Kent State University.

S. M. HENDLEY, M.A. (South Carolina) Engr., Westinghouse Electric Corp., East Pittsburgh, Pa.

P. G. HEYDA, B.S. (London) Math., deHavilland Aircraft Co., Hatfield, Herts, England.

MRS. ALICE A. HUFFMAN, Student, University of California, Riverside.

J. P. JACOB, M.A. (Columbia) Teacher, St. Luke's School, New Canaan, Conn.

W. J. KELSEY, B.A. (Hunter) Junior Actuary, State Insurance Fund, New York, N. Y.

M. J. KEMP, Student, University of Utah.

FRANK LEVIN, Ph.D. (Cincinnati) Asst. Professor, University of Kentucky.

SADIA M. MAKKY, M.S. (Queen Alia C., Iraq) Grad. Student, Indiana University.

H. W. MARTIN, B.A. (West Texas S.C.) Teacher, Clarendon Junior College, Texas.

G. E. MCCUNE, B.A. (Kansas) Pvt., United States Army, Fort Bliss, Texas.

D. S. MCMANUS, B.S. (Columbia) Asst. Professor of Engineering, Norwich University.

E. G. MCNIEL, M.S. (DePaul) Instr., DePaul University.

E. W. MEHNER, B.S. (Case I.T.) Res. Engr., National Cash Register Co., Hawthorne, Calif.

E. L. MESSERE, Student, University of Rhode Island.

C. E. MILLER, Ph.D. (Chicago) Res. Math., California Research Corp., Richmond, Calif.

G. H. PAVLAKOS, M.S. (DePaul) Asst. Professor, Elmhurst College.

GEORGE PEPPERDINE, II, Student, George Pepperdine College.

RONALD PYKE, M.S. (Washington) Res. Asst., University of Washington.

THEODOR RANOV, Dr.Ing. (Technical U., Berlin) Professor of Engineering, University of Buffalo.

L. E. REYNOLDS, M.Ed. (Alfred) Chairman of the Department of Mathematics, State University of New York, Agricultural and Technical Institute, Alfred.

W. C. ROUNDS, Math. Instr., Ford Apprentice Training School, Chicago, Ill.

R. M. SCAMURRA, B.A. (Buffalo) Junior Math., Cornell Aeronautical Lab., Buffalo, N. Y.

S. H. SCHOT, M.A. (Maryland) Res. Asst., University of Maryland.

H. M. SCOFIELD, B.Ch.E. (Cornell) Chemical Engr., Linde Air Products, Buffalo, N. Y.

DONNA J. SEAMAN, B.S. (Washington) Physicist, U. S. Naval Proving Ground, Dahlgren, Va.

PEDRO SEN, B.S.E.E. (Mapua I.T.) Instr., University of Santo Tomas, Manila, Philippines.

SEYMOUR SHERMAN, S.M. (Harvard) Staff Member, Dian Labs., New York, N. Y.

E. E. SHULT, B.A. (Southern Illinois) Res. Asst., Biological Research Lab., Southern Illinois University.

R. S. SPIRA, Student, University of California, Berkeley.

THOMAS THEOCLITUS, B.S. (Ohio U.) Dynam-

ics Engr., Bell Aircraft Corp., Niagara Falls, N. Y.

G. P. TRICOLES, A.B. (U.C.L.A.) Res. Engr., Convair Division, General Dynamics Corp., San Diego, Calif.

MRS. MARY K. TULOCK, M.A. (George Peabody) Asst. Professor, California State Polytechnic College.

L. A. WELLER, M.A. (Toronto) Physicist, Foster Wheeler Corp., New York, N. Y.

THE MAY MEETING OF THE WISCONSIN SECTION

The twenty-fourth annual meeting of the Wisconsin Section of the Mathematical Association of America was held at Marquette University, Milwaukee, Wisconsin, on May 7, 1956. Professor A. C. Moeller, Chairman, presided. There were 66 present, including 41 members.

The following officers were elected for the coming year: Chairman, Mr. C. J. Vanderlin, Jr., Wisconsin State College, Whitewater; Vice-Chairman, Mr. Joseph Kennedy, Wisconsin High School, Madison; Secretary-Treasurer, Sister Mary Felice, Mount Mary College, Milwaukee.

The following papers were presented:

1. *Probability and statistics in general education*, by Professor J. V. Talacko, Marquette University, Milwaukee.

Mathematical probability and statistics are an important part of general education. Statistics is a coherent, uniform, mathematical science based on the laws of probability. Institutions of higher learning should have in their curriculum courses in probability and statistics having two objectives: (1) to give students basic instruction in the concepts and methods of statistics, so they may be able to understand statistical reasoning and be able to apply its methods in the fields of their specialization; (2) to train professional, applied or theoretical statisticians.

The number of courses will depend on the size of the institution and on the determination of objectives.

2. *What is a surface?* by Professor L. C. Young, University of Wisconsin, introduced by Professor R. C. Wagner.

The basic notion, originating with Sophus Lie, is that of a weighted set of pairs x, $J(x)$, where x is a point and $J(x)$ determines a positively oriented plane element through x. A surface is defined as such a weighted set subject to an irreducibility condition (akin to connectedness) and a boundary condition. It is derivable from a traditional surface of finite type by an arbitrarily small modification in area. The definition is sufficiently general to ensure, for the first time, the existence of a solution to Plateau's problem without restrictions of topological type. More general weighted sets define "generalized" surfaces.

3. *Geometries*, by Mr. C. J. Vanderlin, Jr., Wisconsin State College, Whitewater.

The paper presented a brief survey of various geometries and their relationships. An outline of Artin's proof that a projective plane can be co-ordinatized over a division ring if and only if Desargue's theorem is satisfied was given. This division ring is commutative if and only if the Pappus-Pascal theorem is satisfied. Following Klein's Erlanger program, affine, unimodular affine, euclidean, metrical, and non-euclidean geometries were presented as sub-geometries of projective geometry.

4. *The Science Teaching Improvement Program of the American Association for the Advancement of Science*, by Dr. J. R. Mayor, Director of Program, Washington, D. C.

Professional scientific organizations are now more deeply concerned about science and mathematics education in the secondary schools than at any other time in this century. Scientists recognize that they have a responsibility for the improvement of secondary-school teaching which has not been adequately met. Throughout the nation there is evidence of new interest in teacher education and of a desire on the part of college and university staff members in academic departments to cooperate with secondary-school teachers and administrators in improving secondary-school programs. If this interest and spirit of cooperation can be properly coordinated and fostered, desirable modification of some goals and trends in secondary education that have been questioned by scientists could result.

5. *Special Mathematics Programs*

a. Report by Professor P. C. Rosenbloom, University of Minnesota.

Since April, 1955, special programs in mathematics have been conducted for superior high school, and even younger, students with special classes weekly, one for high school students, one for junior high, and some for 4th to 6th grade children. Attendance is purely voluntary, but has consistently averaged 40 in the first two groups and 20 in the third. This program was set up with the cooperation of H. O. Jackson, recently appointed consultant in mathematics to the Minneapolis public school system, and F. Roessel, principal of Lincoln Junior High School. The importance of having people in the school systems with primary responsibility for the gifted students was stressed.

b. Report by Mr. Joseph Kennedy, Wisconsin High School, Madison.

Mr. Kennedy described briefly an experimental twelfth grade course at Wisconsin High School. This course included a number of topics from modern mathematics; the text was *Principles of Mathematics* by Allendoerfer and Oakley. It is difficult to draw conclusions from one year's work, but it appeared that students grasped abstract ideas easily, their approach to problems became less inhibited, they displayed knowledge of important concepts, and they found the course an enjoyable experience. This teacher believes that almost any twelfth grade class would profit from a course based on this text, although some topics would have to be omitted for a less able group.

7. *A universal metric for experimental sciences, Biological time and gamma function, Elementary problems in theory and elasticity, On the geometry of* $\Omega = z^2$, by Professor Jose Gallego-Diaz, Special School of Agricultural Engineering, Madrid, Spain. (Presented by title.)

SISTER MARY FELICE, *Secretary*

ORGANIZATIONAL MEETING OF THE NEW JERSEY SECTION

At a meeting of New Jersey mathematicians to be held November 3, 1956, at Rutgers University a motion to organize a New Jersey section of the Association will be entertained.

Dean A. E. Meder, Jr., is temporary chairman; Professor C. A. Nelson is chairman of the committee on by-laws; Professor I. L. Battin is chairman of the nominating committee; and Professor D. R. Davis is chairman of the program committee.

I. L. BATTIN, *Temporary Secretary*

CALENDAR OF FUTURE MEETINGS

Fortieth Annual Meeting, University of Rochester, Rochester, New York, December 29, 1956.

Thirty-eighth Summer Meeting, Pennsylvania State University, University Park, Pennsylvania, August 26–27, 1957.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Associate Secretary.

ALLEGHENY MOUNTAIN, Westinghouse Research Laboratories, Pittsburgh, Pennsylvania, Spring, 1957.

ILLINOIS, Illinois State Normal University, Normal, May 10–11, 1957.

INDIANA, May 4, 1957.

IOWA, Iowa State Teachers College, Cedar Falls, April 26–27, 1957.

KANSAS

KENTUCKY, Berea College, Berea, April 20, 1957.

LOUISIANA-MISSISSIPPI, Buena Vista Hotel, Biloxi, Mississippi, February 15–16, 1957.

MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, College of William and Mary, Williamsburg, Virginia, December 1, 1956.

METROPOLITAN NEW YORK

MICHIGAN, Wayne University, Detroit, March 23, 1957.

MINNESOTA

MISSOURI, Southeast Missouri State College, Cape Girardeau, Spring, 1957.

NEBRASKA, University of Nebraska, Lincoln, April 27, 1957.

NEW JERSEY, Organizational Meeting, Rutgers University, November 3, 1956.

NORTHEASTERN, University of Connecticut, Storrs, November 24, 1956.

NORTHERN CALIFORNIA, University of California, Berkeley, January 12, 1957.

OHIO

OKLAHOMA

PACIFIC NORTHWEST, Oregon State College, Corvallis, June, 1957.

PHILADELPHIA, Muhlenberg College, Allentown, Pennsylvania, November 24, 1956.

ROCKY MOUNTAIN, Colorado School of Mines, Golden, Spring, 1957.

SOUTHEASTERN, Emory University, Emory University, Georgia, March 15–16, 1957.

SOUTHERN CALIFORNIA, San Diego State College, Spring, 1957.

SOUTHWESTERN, University of Arizona, Tucson, April, 1957.

TEXAS, University of Houston, Houston, April, 1957.

UPPER NEW YORK STATE, Skidmore College, Saratoga Springs, May 4, 1957.

WISCONSIN, Wisconsin State College, Whitewater, May, 1957.

THE CARUS MATHEMATICAL MONOGRAPHS

These Monographs are a series of expository books intended to make topics in pure and applied mathematics accessible to teachers and students of mathematics and also to non-specialists and scientific workers in other fields. One copy of each Monograph may be purchased by members of the Association for $1.75 each. Additional copies and copies for non-members of Monographs 1–8 are priced at $3.00 each, and must be purchased from the Open Court Publishing Co., LaSalle, Illinois. In the case of Monographs 9, 10 and 11 additional copies and copies for non-members must be purchased at $3.00 from John Wiley and Sons, 440 Fourth Ave., New York 16, N. Y. The most recent issues are:

No. 4. *Projective Geometry* by J. W. Young, ix +185 pages.

No. 5. *History of Mathematics in America before 1900* by D. E. Smith and Jekuthiel Ginsburg, viii+210 pages.

No. 6. *Fourier Series and Orthogonal Polynomials* by Dunham Jackson, xiv+234 pages.

No. 7. *Vectors and Matrices* by C. C. MacDuffee, xi+192 pages.

No. 8. *Rings and Ideals* by N. H. McCoy, xii +216 pages.

No. 9. *The Theory of Algebraic Numbers* by Harry Pollard, xii+143 pages.

No. 10. *The Arithmetic Theory of Quadratic Forms* by B. W. Jones, x+212 pages.

No. 11. *Irrational Numbers* by Ivan Niven, xii+164 pages.

METALS RESEARCH LABORATORIES

ELECTRO METALLURGICAL COMPANY

Requires .

OPERATIONS RESEARCH ANALYST
RESEARCH MATHEMATICIAN

To work in a closely knit operations research type group of an expanding metallurgical and process research laboratory. Studies to be made of the allocation of resources and effort among various research projects. Evaluate new areas of interest and examine various operations of both Laboratories and Company to determine fields in which operations research might be fruitfully applied. High speed computing equipment available.

OPERATIONS RESEARCH ANALYST: Ph.D. in operations research or equivalent with M.S. or B.S. in Engineering, Physical Science, or Mathematics. To set up mathematical models for both economic and physical processes. Linear, quadratic and dynamic programming as well as inventory smoothing and statistical prediction and other techniques. No industrial experience necessary.

RESEARCH MATHEMATICIAN: Ph.D. in Mathematical Physics or applied mathematics with B.S. in Physics or related discipline. To set up mathematical models for computation, assist experimentalists in setting up data reduction schemes for computer or in other methods of analyzing experimental results mathematically as well as in setting up experimental designs. Originate the application of new techniques for these purposes. Several years of experience in applied mathematics, mathmatical physics or operations research desirable but not essential for the right man. Please send resume including salary desired to J. L. Lamont, Personnel Administrator, Metals Research Laboratories, Electro Metallurgical Company, P.O. Box 580, Niagara Falls, New York.

COMPUTER PROGRAMMERS

B.S. M.S. Ph.D.

Opportunities in Florida's Playground Area

UNIVAC SCIENTIFIC .....IBM 650

DATATRON .....IBM 704

If you are interested in improving your living and working conditions you are invited to investigate the opportunities afforded by Vitro in Florida's Playground Area. The Playground Area is located on the Gulf Coast of Florida at Fort Walton Beach. It is primarily a summer resort although winters are mild with temperatures seldom descending to the freezing level. All of the usual beach water sports such as water skiing, boating, swimming and fishing are prevalent. This area also offers excellent hunting and boasts a fine golf course. The Playground Area has a population of approximately 25,000.

Programmers who are interested in diversified and challenging assignments evaluating the latest developments in rockets, bombs, fire control and bombing systems, produced by industry, are invited to contact:

VITRO LABORATORIES
EGLIN AIR FORCE BASE
FLORIDA

Similar positions also available at Vitro's Laboratories at West Orange, New Jersey, and Silver Spring, Maryland.

MATHEMATICIANS

For Analysis Group of expanding research and development laboratory. Principal fields of interest are: weapons systems analysis, peacetime applications of atomic energy, and operations research. Several openings are available.

1 To carry out studies in OPERATIONS RESEARCH. Familiarity with probability theory, linear programming, game theory, information theory, optimization procedures, and other OR techniques very desirable.

2 To perform operational analyses requiring extensive background in aerodynamics and exterior ballistics. May also investigate problems dealing with missile fire control and navigational systems. Familiarity with digital computer programming desirable.

3 To conduct investigations in the fields of electromagnetic theory, acoustics, thermal and radiation effects.

These openings require men with vision and initiative. Our modern laboratory provides a professional working atmosphere and the location in a quiet suburban area provides pleasant living and working with easy access to the cultural and educational facilities of New York City.

All inquiries in confidence. Please send resume, including salary desired, to Personnel Manager.

VITRO LABORATORIES

Division of Vitro Corporation of America

200 Pleasant Valley Way
West Orange, New Jersey

SYSTEMS ENGINEERING

Openings now exist at BATTELLE in the following areas:

MATHEMATICAL and STATISTICAL ANALYSIS

DIGITAL and ANALOG COMPUTER PROGRAMMING and PROBLEMS ANALYSIS

THEORETICAL PHYSICS

NUCLEAR REACTOR POWER SYSTEM CONTROL

CONTROL SYSTEMS ANALYSIS

Employment on Battelle's Systems Engineering staff assures an opportunity to engage in an activity of a professional nature. Advanced degrees are obtainable, at no cost to the staff member, through BATTELLE'S TECHNICAL EDUCATIONAL PROGRAM. OUTSTANDING BENEFIT PROGRAM including FOUR WEEKS VACATION after only five years employment offered by this research organization employing 3,000, two-thirds of whom are engaged directly in the research effort. For descriptive material and technical application form, write to

Russell S. Drum
Technical Personnel Manager
BATTELLE INSTITUTE
505 King Avenue, Columbus 1, Ohio

The Second in a Series of Announcements on Progressive Expansion of Program and Facilities in Mathematics at the Knolls Atomic Power Laboratory:

GENERAL ELECTRIC'S
KNOLLS ATOMIC POWER LABORATORY

Announces IT IS NOW DOUBLING THE STAFF OF MATHEMATICIANS FOR ITS MODERN MATHEMATICAL CENTER

The steadily advancing nuclear program at Knolls Atomic Power Laboratory calls for new and imaginative departures in mathematics—ranging from the most abstruse formulations of fundamental problems to the digital solution of physical problems. To meet the consequent expansion of its Mathematical Analysis Program, the Laboratory plans to increase significantly the number of qualified mathematicians now at work here—enough new openings have been created, in fact, to more than double the present mathematical staff. Mathematicians at all degree levels are invited to join this expanding program.

As previously announced, a modern building is now under construction, principally for the use of mathematicians and physicists. This Center will be equipped with the finest of facilities, including digital computers that rank among the most powerful available. Here mathematicians, working both independently and in association with theoretical and experimental physicists, will enjoy an atmosphere in which the creative mind may find its full fruition.

As members of the Mathematical Analysis Unit, they will participate in the formulation of theories to describe new physical situations now being encountered, in evaluating these theories and adapting them to numerical solution by digital computers, and in evaluating reactor designs. Design evaluations will focus on the calculated behavior of mathematical models and will employ the most modern techniques in computer programming. The nature and complexity of these operations call for creatively new approaches and fundamental advances in these techniques. These mathematicians will also have the opportunity to deal with basic research in physics, chemistry, metallurgy and many other aspects of nuclear science.

The program at the Knolls offers the atmosphere, the equipment, the richness of subject matter and the material benefits conducive to a satisfying career in applications of mathematics.

A LETTER TO DR. S. R. ACKER, EXPRESSING YOUR INTEREST, WILL RECEIVE IMMEDIATE ATTENTION.

Knolls Atomic Power Laboratory

OPERATED FOR A.E.C. BY

GENERAL ELECTRIC

SCHENECTADY, N. Y.

WILEY — BOOKS

PRINCIPLES AND TECHNIQUES OF APPLIED MATHEMATICS

By BERNARD FRIEDMAN, *New York University.* This work shows how the study of abstract systems can provide methods for the solution of concrete problems. A feature of the book is its explanation of how *explicit* solutions of partial differential equations can be obtained. The work also covers the Schwartz theory of distributions and the Fredholm theory of integral equations. A simple proof for the reduction of a matrix to Jordan canonical form, using generalized eigenvectors, is provided. A publication in Wiley's Applied Mathematics Series, edited by I. S. Sokolnikoff. *1956. 315 pages. $8.00.*

IRRATIONAL NUMBERS

By IVAN NIVEN, *University of Oregon.* Number 11 in the series of Carus Mathematical Monographs. The book offers an exposition of some central results on irrational numbers. The main emphasis is on those aspects of irrational numbers commonly associated with number theory and Diophantine approximations. Complete treatment is given irrational numbers, normal numbers, and the basic theorems on transcendental numbers. *1956. 164 pages. $3.00.*

AUTOMATIC DIGITAL COMPUTERS

By M. V. WILKES, *Cambridge University Mathematical Laboratory.* Written to furnish a general introduction to principles underlying both design and use of digital computers. The subject is set in perspective by a historical introduction beginning with Babbage's "difference engine", and then coming down to present-day electronic machines. The principles of logical design are presented next. These are followed by discussions of principles of program construction, relay computers, storage, and electronic switching and computing circuits. The final chapter treats the design and operation of digital computers. *1956. 305 pages. $7.00.*

SYMPOSIUM ON MONTE CARLO METHODS

Edited by H. A. MEYER, *University of Florida.* Papers by twenty-two leading workers in the field, written about their own research and applications. A bibliography with abstracts provides a convenient guide to existing published works on the Monte Carlo method. One of the Wiley Publications in Statistics, *Walter A. Shewhart and S. S. Wilks, editors. 1956. 382 pages. $7.50.*

Send for examination copies

JOHN WILEY & SONS, INC.

440 Fourth Avenue — New York 16, N.Y.

Just published

The outstandingly complete, up-to-date text for today's engineering students

ENGINEERING MATHEMATICS

by Kenneth S. Miller

417 pp., $6.50

Written by a mathematician widely experienced in current engineering research, this modern, thorough treatment of differential equations, network theory and random functions includes much material needed in today's engineering developmental work but heretofore not appearing in similar textbooks. There is, for instance, full treatment of the Frobenius method, the Hermite, Laguerre and Tschebyscheff functions, Green's function, Wiener-Khintchine relations, and the use of stochastic processes in engineering probability. Appendices contain Borel Sets, Fourier Integrals and the Riemann-Stieltjes Integral.

From the publishers of

A growing list of distinguished texts under the expert editorship of

CARROLL V. NEWSOM

newly elected President of New York University

The new

CALCULUS *by Jack R. Britton*

584 pp., $6.50

This unusually clear and teachable presentation of the differential and integral calculus, with its abundance of excellent problems, is already being widely welcomed by teachers across the country

Introduction to Modern Algebra and Matrix Theory
by R. A. Beaumont & R. W. Ball $6.00

Algebra for College Students
REVISED EDITION by Jack R. Britton & L. C. Snively $4.50

College Algebra
by Jack R. Britton & L. C. Snively $5.00

Intermediate Algebra
by Jack R. Britton & L. C. Snively $3.00

An Introduction to the History of Mathematics
by Howard Eves $6.00

Real Functions
by Casper Goffman $6.00

Fundamentals of College Mathematics
by R. E. Johnson, N. H. McCoy & A. F. O'Neill $5.00

The Theory of Numbers
by Burton W. Jones $3.75

Analytic Geometry
by N. H. McCoy & R. E. Johnson $3.50

Freshman Mathematics.
THIRD EDITION by Carroll V. Newsom $5.50

Mathematics of Investment
by Paul R. Rider & C. H. Fischer $5.00

Rinehart Mathematical Tables, Formulas and Curves
ENLARGED EDITION by Harold Larsen $2.50

RINEHART
& Company, Incorporated **232 Madison Ave., New York 16**

authoritative, modern texts

COLLEGE ALGEBRA, REVISED

E. A. Cameron **E. T. Browne**

Maintaining the recognized high caliber of the original edition, this revision generally strengthens material and presents the topics of college algebra with as much rigor as is consistent with sustaining student interest. It features new material on statistics and increased emphasis on applications to physics and chemistry.

INTRODUCTION TO MATHEMATICS, REVISED

Lee Emerson Boyer

Meeting the need for a basic non-technical text, this highly readable book presents the development of mathematics from earliest times to the present. With its unifying historical theme and clear-cut presentation, the book may be used profitably by all students who wish to increase their cultural background and gain a working knowledge of mathematical skills.

Brixey · Andree	MODERN TRIGONOMETRY
Brixey · Andree	FUNDAMENTALS OF COLLEGE MATHEMATICS
Sisam · Atchison	ANALYTIC GEOMETRY, THIRD EDITION
Hill · Linker	INTRODUCTION TO COLLEGE MATHEMATICS, REVISED
E. G. Begle	INTRODUCTORY CALCULUS With Analytic Geometry
G. M. Merriman	CALCULUS

HENRY HOLT AND CO. • 383 Madison • New York 17, N.Y.

WILLIAM L. HART:

Calculus

A thorough, modern presentation, rigorous, yet skillfully adapted to student understanding. Organization, textual exposition, examples, exercises, and problems of widely varied application make this an outstanding text. 639p. (558p. of text).* $5.50

College Algebra, 4th ed.

A tested and favored text with a modern, attractive format. Abundance of good problems and review exercises, logical arrangement of topics, emphasis on modern viewpoints. 490p. (420p. of text).* $4.00

Intermediate Algebra for Colleges

Designed for students who will take a regular college algebra course later, or for students who need algebra as a prerequisite for elementary courses in the social sciences or business administration. 323p. (276p. of text).* $3.75

E. J. CAMP: Mathematical Analysis

A new text containing an integrated treatment of topics from college algebra, trigonometry, analytic geometry, and calculus for the freshman year. 624p. (561p. of text).* $6.00

WILSON and TRACEY:

Analytic Geometry, 3rd ed.

A long-time favorite, in modern format, with new problems and up-to-date applications. 328p.* $4.00

* *Answers for odd-numbered problems included at the back of the book. Answers for even-numbered problems available free.*

D. C. Heath and Company

Sales Offices: Englewood, N.J., Chicago 16, San Francisco 5, Atlanta 3, Dallas 1
Home Office: Boston 16

ANALYTIC GEOMETRY

by PAUL K. REES, Louisiana State University

Here is a well-organized, systematic treatment of analytic geometry which makes teaching and studying the subject simpler by the use of these features:

1. Uses necessary technical terminology, but in a readily understandable manner.
2. Written with awareness of students' difficulty covering material necessary in calculus; uses just 44 exercises.
3. Exercises located so that selections from an exercise and corresponding text constitute a normal assignment.
4. 1,200 problems in groups of 4 so that one of each 4 may be used for complete coverage.
5. Detailed illustrative examples located so they clarify subject matter immediately preceding them.

6″ x 9″ • 256 pages • Published April 1956

MODERN ELEMENTARY STATISTICS

by JOHN E. FREUND, Alfred University

This text is designed for students in the social and natural sciences who have little background in mathematics. It emphasizes the meaning of statistics rather than the acquisition of mathematical skills. Theoretical distributions are introduced as early as Chapter 3 on a more or less intuitive level. Chapter 7 has a discussion and repeated emphasis on the meaning of probability statements. For the first time, the modern theory of the testing of hypotheses is presented on the non-technical level. *FEATURES:* 108 illustrations and exercises covering all the natural and social sciences—two full chapters on business statistics—special chapter on nature of scientific predictions—complete chapter on role of statistics in modern science.

5⅝″ x 8⅜″ • 418 pages • Published 1952

ADVANCED CALCULUS FOR ENGINEERS

by FRANCIS B. HILDEBRAND, Massachusetts Institute of Technology

Here is an integrated presentation of modern developments, special topics and useful methods of calculus essential to engineers and physicists.

Features:

Direct treatment of topics offered wherever possible.

Many applications illustrate the usefulness of concepts and techniques.

Includes sufficient detail to facilitate self-study and thus lessen burden on instructor.

Includes thorough discussion of vector analysis, and explicit applications of complex variables to the evaluation of real definite integrals.

5½″ x 8½″ • 594 pages • Published 1949

For approval copies write

Prentice-Hall, Inc. ENGLEWOOD CLIFFS, NEW JERSEY

CHECK THESE FOUR MACMILLAN TEXTS
for the spring semester

✓ METHODS IN NUMERICAL ANALYSIS

by KAJ L. NIELSEN, United States Naval Ordnance Plant, Indianapolis

This textbook shows the student how to analyze tabulated data and solve equations by numerical methods. Easy to teach from (with illustrative examples and valuable schematics), easy to learn from (clear focus on the essentials), Dr. Nielsen's book describes the methods of solution for the majority of numerical problems encountered by engineers, physicists mathematicians, and statisticians.

1956 382 pp. $6.90

✓ CALCULUS

JOHN F. RANDOLPH, University of Rochester

This text presents a balanced study of the theory and application of calculus. "The material is sufficiently traditional in scope and arrangement to be used in a routine course, but it is more challenging to the better student than are most books on the elementary level. Unusual care is exercised in the presentation of basic concepts, and many topics beyond the usual minimum syllabus are included."—*Scripta Mathematica*

1952 483 pp. $6.00

✓ ELEMENTARY DIFFERENTIAL EQUATIONS

by EARL D. RAINVILLE, University of Michigan

This text provides a complete introduction to elementary differential equations for a full year's course for students who have had calculus. The material is arranged to permit great flexibility in the choice of topics so that it can also be used for a one-semester course. The author offers a sound training in techniques for obtaining solutions and an explanation of the basic theory behind the techniques.

1952 392 pp. $5.00

✓ VECTOR AND TENSOR ANALYSIS

by NATHANIEL COBURN, University of Michigan

This text offers a significant study of vector and tensor analysis and shows the value of these subjects as tools in treating compressible fluids and homogeneous turbulence. Dr. Coburn develops the general theory of vectors and tensors before making application to specific subjects. Although vector analysis is treated in the conventional Gibbs manner, a distributive star product has been introduced to prepare the student for the study of more general quantities.

1955 341 pp. $7.00

The Macmillan Company
60 FIFTH AVENUE, NEW YORK 11, N.Y.

Distributors in Canada
BRETT-MACMILLAN LTD.
25 HOLLINGER ROAD, TORONTO 16

Timely McGRAW-HILL *Books*

INTERMEDIATE ALGEBRA, *New Second Edition*

By PAUL K. REES, Louisiana State University and FRED W. SPARKS, Texas Technological College. 320 pages, $3.90

Designed for the student with only one year of high school algebra, this new edition of a basic text includes the necessary material for further work in mathematics, for the required courses in science, for statistics and for mathematics of finance. The major purpose of the revision has been to provide more and better problems. In addition, many of the illustrative examples have been replaced or rewritten for greater clarity.

COLLEGE ALGEBRA, *New Third Edition*

By PAUL K. REES and FRED W. SPARKS. 455 pages, $4.25

A very thorough and competent revision of a successful algebra text. Fresh, stimulating, and logical, it presents the review that most students need for their college mathematics. The text includes more than the usual number of formal definitions, illustrated by judiciously chosen, carefully explained examples. It reviews such topics as the fundamental operations, special products, factoring, exponents, radicals, linear and fractional equations, and simultaneous linear equations.

PLANE TRIGONOMETRY, *Second Edition*

By E. RICHARD HEINEMAN, Texas Technological College. 256 pages, $3.75 (without tables, $3.25)

Designed especially for students with average mathematical background, this text seeks to establish the habit of logical and independent thinking. Memory work is reduced to a minimum, and all unnecessary formulas and concepts have been omitted. Offers 1274 carefully graded problems. Emphasis in this new edition is placed on the analytic aspects of trigonometry, but complete coverage is given to the solution of triangles.

Alternate Edition, 184 pages, $3.50 (Without tables, $3.00).

Contains 1420 problems, nearly all different from those in the regular edition, thus making available 2694 exercises. The text material is essentially the same as the regular edition.

Send for copies on approval

McGraw-Hill Book Company, Inc.

330 West 42nd Street New York 36, N.Y.

GEORGE BANTA COMPANY, INC., MENASHA, WISCONSIN, U.S.A.

CHNOLOGY DEPT.

THE AMERICAN
MATHEMATICAL MONTHLY

THE OFFICIAL JOURNAL OF
THE MATHEMATICAL ASSOCIATION OF AMERICA, INC.

VOLUME 63 NUMBER 9

PUBLIC LIBRARY
NOV 21 1956
DETROIT

PART II

The Conjugate Coordinate System for Plane Euclidean Geometry

WALTER B. CARVER

Number 5
of the
HERBERT ELLSWORTH SLAUGHT
MEMORIAL PAPERS

NOVEMBER 1956

ONE DOLLAR A COPY

The AMERICAN MATHEMATICAL MONTHLY

(Founded in 1894 by Benjamin F. Finkel)

Carl B. Allendoerfer, *Editor*

Associate Editors

R. V. Churchill	Edwin Hewitt	E. P. Starke
Howard Eves	L. M. Kelly	G. B. Thomas
F. A. Ficken	Ivan Niven	E. P. Vance
Wallace Givens	Edith R. Schneckenburger	H. S. Zuckerman
	George Springer	

EDITORIAL CORRESPONDENCE should be addressed to the Editor, C. B. Allendoerfer, Department of Mathematics, University of Washington, Seattle 5, Washington.

ADVERTISING CORRESPONDENCE should be addressed to R. W. MacDowell, Morey Hall, University of Rochester, Rochester 3, N. Y.

NOTICE OF CHANGE OF ADDRESS by members of the Association as well as correspondence regarding subscriptions to the Monthly should be sent to the Secretary-Treasurer, H. M. Gehman, University of Buffalo, Buffalo 14, N. Y.

THIS IS THE OFFICIAL JOURNAL OF THE

MATHEMATICAL ASSOCIATION OF AMERICA, INC.

(Devoted to the Interests of Collegiate Mathematics)

OFFICERS OF THE ASSOCIATION

President, W. L. Duren, Jr., University of Virginia
First Vice-President, R. V. Churchill, University of Michigan
Second Vice-President, G. B. Price, University of Kansas
Secretary-Treasurer, H. M. Gehman, University of Buffalo
Associate Secretary, Edith R. Schneckenburger, University of Buffalo
Editor, C. B. Allendoerfer, University of Washington

Additional Members of the Board of Governors: M. A. Basoco, H. W. Brinkmann, P. H. Daus, H. H. Downing, P. D. Edwards, N. J. Fine, R. M. Foster, Philip Franklin, F. F. Helton, M. S. Hendrickson, A. S. Householder, R. C. Huffer, C. A. Hutchinson, L. W. Johnson, G. K. Kalisch, E. C. Kiefer, F. W. Kokomoor, R. E. Langer, Z. L. Loflin, L. L. Lowenstein, Saunders MacLane, E. J. McShane, W. H. Myers, Ivan Niven, Morris Ostrofsky, O. J. Ramler, J. F. Randolph, C. B. Read, C. R. Sherer, M. F. Smiley, C. E. Springer, B. M. Stewart, G. B. Thomas, Jr., Bernard Vinograde, M. A. Zorn

---

Entered as second class matter at the post office at Menasha, Wis. Acceptance for mailing at special rate of postage provided for in the Act of February 28, 1925, embodied in Paragraph 4, Section 538, P. L. and R., authorized April 1, 1926.

Subscription Price: To Members, $4 a Year. To Others, $5 a Year.

Published by the Association at Menasha, Wisconsin, and Seattle, Wash., during the months of January, February, March, April, May, June–July, August–September, October, November, December.

Printed in the United States of America

THE CONJUGATE COORDINATE SYSTEM FOR PLANE EUCLIDEAN GEOMETRY

By

WALTER B. CARVER

Professor of Mathematics, Emeritus, Cornell University

The Fifth

HERBERT ELLSWORTH SLAUGHT

MEMORIAL PAPER

This publication has been financed by the Arnold Buffum Chace Fund of the Mathematical Association of America

Published as a supplement to the AMERICAN MATHEMATICAL MONTHLY

Volume 63 Number 9

November, 1956

PREFACE

The author of this paper claims no originality for either the method or the geometric results presented. He is attempting an elementary exposition of a method of handling certain types of problems in plane Euclidean geometry. The method was used by Professor Frank Morley in a number of papers published just before and after the turn of the century, and some explanation of it is given in the second part of his book on *Inversive Geometry*. Morley passed the method on to his students at Johns Hopkins University, and some of them have made use of it in papers appearing in this MONTHLY, the *American Journal of Mathematics* and other periodicals.

The method is, in the opinion of the author, sufficiently simple for presentation to undergraduates with only the first course in calculus and an interest in mathematics as prerequisites; and the material of this paper is based largely on notes for a one-semester course which the author gave several times at Cornell University to mixed classes of undergraduate and graduate students. The paper is addressed to that level of mathematical maturity.

The intention is to present the method rather than geometric results. Most of the latter may be found in books using synthetic methods, such as Altshiller-Court's *College Geometry* or Johnson's *Modern Geometry*. Such books will furnish the interested reader with further geometric theorems the proof of which will be good exercises in the use of conjugate coordinates.

The author wishes to acknowledge the help of Professor Howard Eves, who read the manuscript and made a number of helpful suggestions.

W. B. C.

TABLE OF CONTENTS

Chapter		*Page*
I.	*Complex numbers and their graphical representation*	1
	1.1 Three forms of a complex number	1
	1.2 Points and vectors corresponding to complex numbers	2
	1.3 Conjugate complex numbers	3
	1.4 Products and turns	3
	1.5 Powers and roots	4
	1.6 The logarithm of a turn	5
	1.7 Geometric applications	5
II.	*The conjugate coordinate system.*	8
	2.1 Conjugate coordinates of a point.	8
	2.2 Self-conjugate equations	8
	2.3 The linear self-conjugate equation	9
III.	*The straight line.*	10
	3.1 A map equation and self-conjugate equation of a line	10
	3.2 The clinant and orientation of a line	12
	3.3 The angle from one line to another; perpendicularity	14
	3.4 The line through two points.	15
	3.5 Inverse points with respect to a line	16
IV.	*The circle.*	18
	4.1 A map equation and self-conjugate equation of a circle	18
	4.2 Inverse points with respect to a circle	19
	4.3 Polar points and lines; self-polar triangles.	21
	4.4 The power of a point with respect to a circle; the radical axis of two circles	22
	4.5 Pencils of circles	24
V.	*Map equations and the derivative* $D_t x$	26
	5.1 The equations $x=R(t)$ and $y=\overline{R}(1/t)$	26
	5.2 Rotation, expansion, and translation	26
	5.3 The map equation $x=(a+bt)/(c+dt)$	27
	5.4 The derivatives $D_t x$ and $D_t y$; the clinant of a tangent to a curve	28
	5.5 The derivative $D_z y$	29
VI.	*Symmetric functions of turns*	30
	6.1 The elementary symmetric functions of n turns	30
	6.2 The symmetric functions of $n-1$ out of n turns	30
	6.3 Rational functions of n turns symmetric in $n-1$ of the turns	31

VII. *Some geometry of the triangle* 32
7.1 The circumcenter, centroid, and orthocenter on the Euler line 32
7.2 The nine-point circle and center 33
7.3 Right, obtuse, and acute triangles 35
7.4 The tangential triangle and its circumcircle 35
7.5 The symmedian point and Brocard line 38
7.6 The nine-point circle of the tangential triangle; Feuerbach's theorem 39
7.7 The polar circle of a triangle; a pencil of circles 41
7.8 The circles of Apollonius. 44
7.9 The isodynamic points 47
VIII. *The parabola* 51
8.1 A map equation of a parabola 51
8.2 Tangents to a parabola 53
8.3 Triangles formed by three tangents 53
8.4 The proper 4-line; its centric-circle and node 55
IX. *Hypocycloids* 58
9.1 A map equation of a hypocycloid 58
9.2 The hypocycloid of two cusps 59
9.3 The deltoid; penosculants of a deltoid 60
9.4 Penosculants of a hypocycloid of n cusps 62
9.5 Three tangents to a deltoid 64
9.6 The proper 4-line 66
9.7 Nodal hypocycloids 67
X. *Epicycloids and trochoids* 68
10.1 A map equation of an epicycloid 68
10.2 The cardioid; penosculants of a cardioid 69
10.3 The proper 4-line 72
10.4 A map equation of a trochoid 73
10.5 The ellipse as a trochoid 74
XI. *The rectangular hyperbola* 77
11.1 A map equation and a self-conjugate equation of a rectangular hyperbola 77
11.2 The rectangular hyperbola through four points of the unit circle 78
11.3 The rectangular hyperbolas through the vertices of a triangle 80
11.4 Simson lines and the Simson deltoid of a triangle 83
References 85

Chapter I

COMPLEX NUMBERS AND THEIR GRAPHICAL REPRESENTATION

1.1. We first call the attention of the reader to certain facts, probably already familiar to him, about complex numbers and their graphical representation in the plane.

A complex number a is a number of the form $\alpha+i\beta$, where α and β are real and $i^2=1$. The same complex number may be expressed in two other forms, thus

$$a = \alpha + i\beta = \rho(\cos\theta + i\sin\theta) = \rho e^{i\theta},^*$$

where ρ and θ are real numbers, $\rho \geqq 0$, and $\cos\theta$, $\sin\theta$ mean the cosine and sine of an angle of θ radians; and we have the relations

$$(1.11)\quad \begin{cases} \rho = \sqrt{\alpha^2+\beta^2}, & \cos\theta = \dfrac{\alpha}{\sqrt{\alpha^2+\beta^2}}, \quad \sin\theta = \dfrac{\beta}{\sqrt{\alpha^2+\beta^2}}, \\ \alpha = \rho\cos\theta, & \beta = \rho\sin\theta. \end{cases}$$

If a is zero, $\alpha=\beta=\rho=0$, and θ is any real number whatever. For a given complex number not zero, α, β, and ρ are uniquely fixed, but θ may have any one of an infinite set of values differing from each other by multiples of 2π. We call ρ the *absolute value* (or *modulus*) of a, and θ the *amplitude* of a, and we write

$$\rho = |a|, \qquad \theta = \operatorname{amp} a.$$

If two complex numbers,

$$a_1 = \alpha_1 + i\beta_1 = \rho_1 e^{i\theta_1}$$

and

$$a_2 = \alpha_2 + i\beta_2 = \rho_2 e^{i\theta_2},$$

are equal, we must have

$$(1.12)\qquad \alpha_1 = \alpha_2 \quad \text{and} \quad \beta_1 = \beta_2;$$

and also

$$(1.13)\qquad \begin{array}{ll} \text{either} & \rho_1 = \rho_2 = 0, \\ \text{or} & \rho_1 = \rho_2 \neq 0 \end{array}$$

with θ_1 and θ_2 equal or differing only by a multiple of 2π.

* The relation

$$e^{i\theta} = \cos\theta + i\sin\theta$$

may be regarded as the definition of $e^{i\theta}$. See D. R. Curtiss: Analytic Functions of a Complex Variable, p. 51; and Goursat-Hedrick: Mathematical Analysis, vol. II, part I, p. 25.

1.2. A one-to-one correspondence is established between complex numbers and the points of a plane as follows. Let a polar coordinate system be superposed on a rectangular Cartesian system with the polar axis coinciding with the posi-

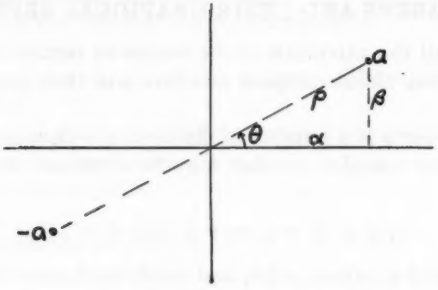

FIG. 1

tive end of the x-axis. Then the complex number a corresponds to the point whose Cartesian coordinates are (α, β) and polar coordinates are (ρ, θ), θ in radians. We may then speak of the *point* a, meaning the point corresponding to

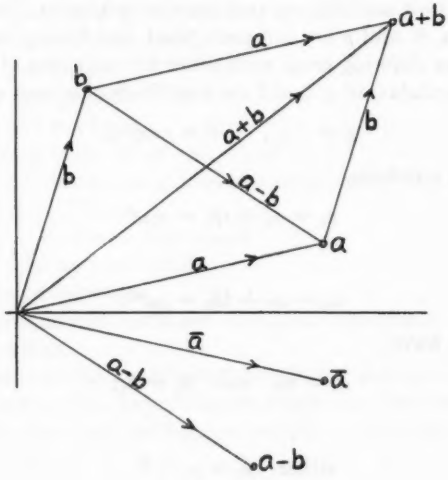

FIG. 2

the complex number a in the above sense. The points a and $-a$ are symmetric with respect to the origin.

We shall also make use of a correspondence between complex numbers and *vectors* in the plane. The complex number a corresponds to the vector having

the length and direction of the vector from the origin to the point a. (We use *vector* in the sense of a *free vector*, the vector being uniquely determined by its length and direction without regard to the position of its initial point.) Because of this correspondence we may speak of the *vector* a. The usual graphical addition and subtraction of vectors will correspond to the algebraic addition and subtraction of the complex numbers. Frequent use will be made of the fact that the vector from the *point* b to the *point* a is the vector $(a-b)$.

1.3. The notation $\bar{a}$ will be used to indicate the complex number *conjugate* to a, *i.e.*,

$$\text{if} \quad a = \alpha + i\beta = \rho e^{i\theta},$$

$$\text{then} \quad \bar{a} = \alpha - i\beta = \rho e^{-i\theta}.$$

The point $\bar{a}$ is the reflection of the point a in the x-axis. The sum and the product of two conjugate complex numbers, $a+\bar{a}$ and $a\bar{a}$, are always real, and $a\bar{a} \geqq 0$. The real numbers α, β, ρ and θ may be expressed in terms of a and $\bar{a}$ as follows:

$$\alpha = \frac{1}{2}(a + \bar{a}), \qquad \beta = \frac{1}{2i}(a - \bar{a}),$$

$$\rho = \sqrt{a\bar{a}}, \qquad \theta = -i \log \frac{a}{\sqrt{a\bar{a}}}. \tag{1.31}$$

If $a=\bar{a}$, $\beta=0$ and a is real.

If $R(z)$ is a rational function of z, $\overline{R}(z)$ will indicate the function obtained by replacing each coefficient in $R(z)$ by its conjugate. Then the conjugate of $R(a)$ is $\overline{R}(\bar{a})$.

1.4. If k is real and not zero, the direction of the vector ka is the same as, or opposite to, that of the vector a according as k is positive or negative; and the length of the vector ka is $|k|$ times that of the vector a.

If the complex number b is multiplied by the complex number $a=\rho e^{i\theta}$, the vector ab corresponding to the product is obtained graphically by turning the vector b through an angle θ and stretching (or contracting) it to ρ times its original length. If $\rho=1$, so that $a=e^{i\theta}$, the vector ab is obtained by merely turning the vector b without changing its length. For this reason a complex number of the form $e^{i\theta}$, with absolute value unity, will be called a *turn*, and will usually be represented by the letter t. A turn corresponds to a point on the unit circle or to a vector of unit length. Since $\sqrt{t\bar{t}}=1$, $\bar{t}=1/t$, *i.e.*, the conjugate of a turn is its reciprocal; and, conversely, if $\bar{a}=1/a$, then a and $\bar{a}$ are turns. The product of two or more turns is a turn, and if t is a turn then t^n is a turn for any real n. The quotient of any number by its conjugate is a turn, $a/\bar{a}=t$, and $\text{amp } t = 2 \text{ amp } a$.

1.5. If $t=e^{i\theta}=\cos\theta+i\sin\theta$ and m is real, then

$$t^m = (e^{i\theta})^m = e^{i(m\theta)} = \cos m\theta + i\sin m\theta.* \tag{1.51}$$

Putting $m=1/n$, n an integer greater than 1, we have

$$t^{1/n} = (e^{i\theta})^{1/n} = e^{i\theta/n},$$

and for a particular value of θ this gives *one* of the nth roots of t. But if we write, as we may,

$$t = e^{i(\theta+2\pi s)}, \qquad s \text{ any integer},$$

we have

$$t^{1/n} = e^{i(\theta+2\pi s)/n},$$

and for $s=0, 1, 2, \cdots, n-1$, this gives the n *different* nth roots of t, all of them turns. The graphical representation of these n roots is interesting. The turn t corresponds to a point on the unit circle with amplitude θ, $0<\theta\leqq 2\pi$. The point on the unit circle with amplitude θ/n will correspond to one of the nth roots of t.

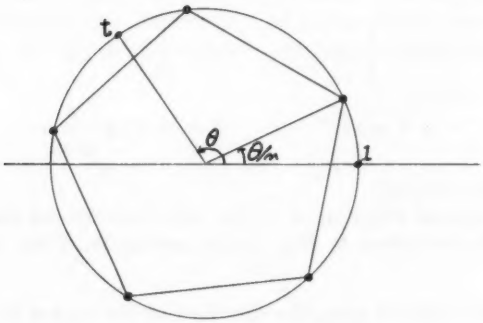

FIG. 3

With this point as one vertex, a regular polygon of n sides may be inscribed in the circle, and the vertices of this polygon will correspond to the n different nth roots of t.

If $t=1$ we have the important case of the nth roots of unity, roots of the equation $z^n=1$, represented by the n vertices of a regular n-gon with one vertex at the point 1. If $\epsilon=e^{2\pi i/n}$, the n different nth roots of unity are

$$\epsilon, \epsilon^2, \epsilon^3, \cdots, \epsilon^{n-1}, \epsilon^n, \qquad (\epsilon^n = 1)$$

* The trigonometric relation

$$(\cos\theta+i\sin\theta)^m=\cos m\theta+i\sin m\theta$$

for m a positive integer is known as De Moivre's theorem, and is found in many textbooks in College Algebra and Trigonometry.

and the sum of these n roots is zero. Certain relations among them are quite obvious either algebraically or graphically. For instance

$$\epsilon^{k+n} = \epsilon^k$$

$$\frac{1}{\epsilon^k} = \overline{\epsilon^k} = \epsilon^{n-k}.$$

If n is even, $\epsilon^{n/2} = -1$, and $-\epsilon^k = \epsilon^{k+n/2}$.

We shall make considerable use of the three cube roots of unity,

$$1, \qquad \omega = e^{2\pi i/3}, \qquad \omega^2 = e^{4\pi i/3}$$

which satisfy the relations

$$\begin{cases} 1 + \omega + \omega^2 = 0, \\ \overline{\omega} = \dfrac{1}{\omega} = \omega^2, \quad \overline{\omega^2} = \dfrac{1}{\omega^2} = \omega. \end{cases} \tag{1.52}$$

The n different nth roots (n a positive integer) of any complex number $a = \rho e^{i\theta}$ are, similarly,

$$\sqrt[n]{\rho}\, e^{i(\theta+2\pi s)/n}, \qquad s = 0, 1, 2, \cdots, n-1,$$

where $\sqrt[n]{\rho}$ means the positive nth root of the positive number ρ; and these n roots correspond to the vertices of a regular n-gon inscribed in a circle of radius $\sqrt[n]{\rho}$ with center at the origin.

1.6. To define the (natural) logarithm of a complex number we agree that when $z = e^u$ then $u = \log z$. Then since

$$t = e^{i(\theta+2\pi s)}, \qquad s \text{ any integer},$$

we have $\log t = i(\theta + 2\pi s)$. Thus, for a fixed t, $\log t$ has infinitely many values differing from each other by multiples of $2\pi i$, and $2\pi i s$, for s any integer, is a logarithm of 1.

1.7. We give some simple examples to show how methods using complex numbers and vectors may be useful in handling geometric problems in the plane.

Two points A and B (Figure 4) are given by the complex numbers a and b; and we wish to find the complex number c corresponding to the point C on the segment AB which divides this segment in the given ratio $m:n$, m and n *positive* numbers.

The vectors from A to C and from C to B correspond respectively to the numbers $c-a$ and $b-c$. Then since these vectors have the same direction (see section 1.4), we have

$$c - a = \frac{m}{n}(b - c)$$

$$nc - na = mb - mc$$

$$(m + n)c = mb + na$$

and

$$c = \frac{mb + na}{m + n}. \tag{1.71}$$

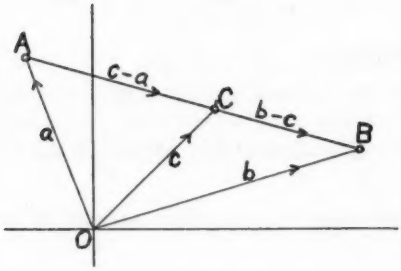

FIG. 4

As a particular case, the point

$$c = \frac{a + b}{2} \tag{1.72}$$

is the *mid-point* of the segment AB.

Using 1.72 and 1.71 one proves readily that if points a, b, c are the vertices of a triangle the point $(a+b+c)/3$ is the *centroid* of the triangle (center of gravity, point of intersection of the medians).

As another example, let us find the condition on the complex numbers a, b, c such that the triangle with vertices at the three corresponding points may be an equilatera! triangle with the vertices a, b, c in counterclockwise order.

It is evidently necessary and sufficient that the vector $a-c$ shall be equal to the vector $c-b$ turned through an angle $2\pi/3$; that is

$$a - c = e^{2\pi i/3}(c - b),$$

or

$$a - c = \omega(c - b),$$

and we have

$$a + \omega b - (1 + \omega)c = 0,$$

or (see 1.52)

$$a + \omega b + \omega^2 c = 0, \tag{1.73}$$

as the required condition.

Similarly,

$$a + \omega^2 b + \omega c = 0 \tag{1.74}$$

is the condition that the points a, b, c should be the vertices of an equilateral triangle in clockwise order.

The reader may use 1.73 to prove

THEOREM 1.75. *If on the sides of any triangle, equilateral triangles are constructed outwardly, the centers of the equilateral triangles will be the vertices of an equilateral triangle.*

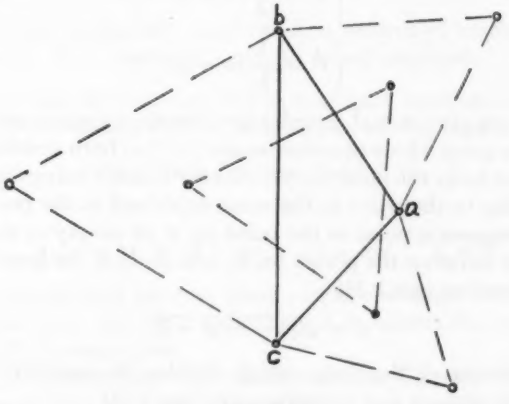

FIG. 5

Chapter II

THE CONJUGATE COORDINATE SYSTEM

2.1. The geometry we shall study will be the *real* Euclidean geometry in the Euclidean plane. A rectangular Cartesian coordinate system is first set up, the Cartesian coordinates of any point in the plane being a pair of *real* numbers, (X, Y). The *conjugate* (or *circular*) *coordinates*, (x, y), of any point are then defined by the transformation

$$\begin{cases} x = X + iY \\ y = X - iY \end{cases}, \tag{2.11}$$

and we have the inverse transformation

$$\begin{cases} X = \dfrac{1}{2}(x + y) \\ Y = \dfrac{1}{2i}(x - y) \end{cases}. \tag{2.12}$$

Since X and Y are always real, x and y are always conjugate complex numbers. The origin is the point whose coordinates are $(0, 0)$ in both systems. Either x or y alone is sufficient to fix the point (x, y), the coordinate x being the complex number corresponding to the point in the sense explained in the preceding chapter; and we may designate a point as the point (x, y) or simply as the point x.

The distance between the points $(a, \bar{a})$ and $(b, \bar{b})$ is the length of the vector $a-b$, and is therefore (see 1.31)

$$d = \sqrt{(a - b)(\bar{a} - \bar{b})}. \tag{2.13}$$

Also the coordinates of the point which divides the segment from $(a, \bar{a})$ to $(b, \bar{b})$ in the ratio $m:n$, m and n positive, are (see 1.71)

$$\left(\frac{na + mb}{m + n}, \frac{n\bar{a} + m\bar{b}}{m + n}\right). \tag{2.14}$$

2.2. We shall be interested in certain algebraic curves, curves which, in the Cartesian coordinate system, have equations of the form

$$P(X, Y) = 0, \tag{2.21}$$

where $P(X, Y)$ is a proper polynomial* in X and Y with *real* coefficients. From the form of the transformations 2.11 and 2.12 it is evident that such a curve will have an equation in the conjugate coordinate system,

$$p(x, y) = 0, \tag{2.22}$$

where $p(x, y)$ is a proper polynomial of the same degree as $P(X, Y)$ but with

* We use the phrase "proper polynomial" in the sense of a polynomial *of degree greater than zero*, so that a constant (which is a polynomial of degree zero) is not a *proper* polynomial.

complex coefficients. But we will show that a proper polynomial equation in x and y does not correspond to a curve unless it satisfies certain conditions.

We assume for the moment that the polynomial $p(x, y)$ is irreducible, *i.e.*, that it is not the product of any two proper polynomials. If the equation $p(x, y)=0$ represents a curve it must be satisfied by infinitely many *conjugate* pairs of values x_1, y_1, and if $p(x_1, y_1)=0$ then also $\bar{p}(y_1, x_1)=0$. Thus the two equations $p(x, y)=0$ and $\bar{p}(y, x)=0$ would have infinitely many common solutions, and the polynomials $p(x, y)$ and $\bar{p}(y, x)$ would have a common proper polynomial factor.* But since $p(x, y)$, and therefore also $\bar{p}(y, x)$, is irreducible, it follows that these polynomials are identical except for a possible constant factor, *i.e.*,

$$\bar{p}(y, x) \equiv \lambda p(x, y), \qquad \lambda \text{ a constant} \neq 0. \tag{2.23}$$

When this condition is satisfied we say that $p(x, y)$ is a *self-conjugate polynomial* and that $p(x, y)=0$ is a *self-conjugate equation*. We have then shown that:

If $p(x, y)$ is an irreducible† polynomial, a necessary‡ condition that $p(x, y)=0$ be the equation of a curve is that $p(x, y)$ be self-conjugate.

2.3. The equation of a straight line in conjugate coordinates will obviously be linear, $ax+by+c=0$; and since it must also be self-conjugate we must have

$$\bar{a}y + \bar{b}x + \bar{c} \equiv \lambda(ax + by + c),$$

or

$$\bar{a} = \lambda b, \qquad \bar{b} = \lambda a, \qquad \bar{c} = \lambda c, \qquad \lambda \neq 0. \tag{2.31}$$

Since a and b cannot both be zero, these conditions imply that *neither a nor b* can be zero. Also, since $\bar{a}/\bar{b}=b/a$ and $\bar{c}/\bar{a}=c/b$, it follows that a/b is a turn and that c/a and c/b are conjugates.

Conversely, it is easily seen (by transforming to Cartesian coordinates) that the equation $ax+by+c=0$ always corresponds to a straight line when conditions 2.31 are satisfied.

The equations $x=0$ and $y=0$ are not self-conjugate and hence do not represent lines, and we do not speak of an x-axis or y-axis. Instead we make use of the *axis of reals* with the equation $x-y=0$, and the *axis of pure imaginaries* with the equation $x+y=0$, and these are respectively the X-axis and Y-axis of the Cartesian system.

---

* Bôcher, Introduction to Higher Algebra, p. 211.

† If $p(x, y)$ is reducible it is not necessary that it be self-conjugate but only that one (or more) of its factors be self-conjugate. Thus $2ix^2-xy+iy^2=(2ix+y)(x+iy)$ is not self-conjugate but the factor $x+iy$ is, and $x+iy=0$ is the equation of a line every point of which satisfies the equation $2ix^2-xy+iy^2=0$.

‡ The condition is not sufficient except for polynomials of the first degree. Thus

$$2xy - ix + iy + 4 = 0$$

is a self-conjugate equation, but it is not satisfied by any pair of *conjugate* numbers x, y, and hence does not represent a curve.

Chapter III

THE STRAIGHT LINE

3.1. A different approach to the straight line will illustrate methods to be used later in the treatment of other curves.

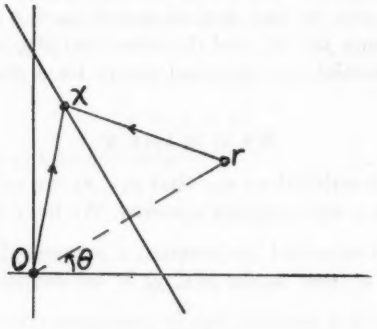

FIG. 6

By the *reflex point* of a line we shall mean the reflection of the origin in the line. If the line goes through the origin, the reflex point is the origin itself. Let r be the reflex point of a line not through the origin, $r=\rho e^{i\theta}$, $\rho\neq 0$, and let x be a variable point on the line. Then the vectors x and $x-r$ are of the same length, and we have

$$x - r = tx,$$

where t is a turn running through all turn values (except the value 1) as the point x runs along the line. Solving for x we have

$$x = \frac{r}{1-t}. \tag{3.11}$$

Such an equation, expressing the x coordinate of a variable point on a curve as a function of a variable *turn* t, will be called a *map equation* of the curve. If x is equal to $r/(1-t)$, then the conjugate of x must be equal to the conjugate of $r/(1-t)$; in other words, the map equation 3.11 implies the conjugate equation

$$y = \frac{\bar{r}}{1-\frac{1}{t}} = \frac{t\bar{r}}{t-1}, \tag{3.12}$$

and 3.11 and 3.12 together give us a pair of parametric equations of the line.

Eliminating the parameter t from the two equations, we obtain a self-conjugate equation of the line,

$$\bar{r}x + ry = r\bar{r}.$$

Since $r \neq 0$, we may divide by r, obtaining

$$\frac{\bar{r}}{r}x + y = \bar{r}.$$

Now $\bar{r}/r(=e^{-2i\theta})$ is a fixed turn τ, and $\bar{r}=\tau r$, and hence a self-conjugate equation of the line is obtained in the standard form

$$\tau\bar{x} + y = \bar{\tau}r. \tag{3.13}$$

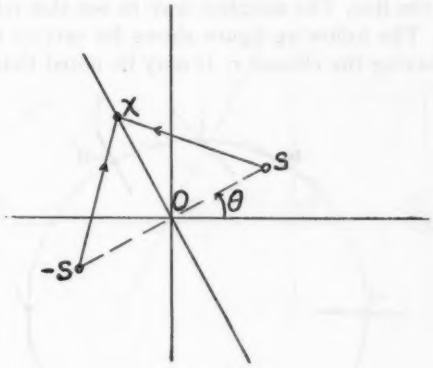

FIG. 7

If the line passes through the origin, $r=0$, and certain steps in the above procedure would not be valid. In this case let s be any point on the perpendicular to the line at the origin, $s=\rho e^{i\theta}$, $\rho \neq 0$. We then have

$$x - s = t(x + s),$$

or

$$x = \frac{s(1+t)}{1-t}, \tag{3.14}$$

as a map equation of the line. This implies the conjugate equation

$$y = \frac{s\left(1+\frac{1}{t}\right)}{1-\frac{1}{t}} = \frac{s(t+1)}{t-1},$$

and eliminating t as before we have the self-conjugate equation

$$\bar{s}x + sy = 0.$$

Letting $\bar{s}/s = \tau$, this becomes

$$\tau x + y = 0,$$

which agrees with 3.13 when $\tau = 0$. Hence equation 3.13 is a general standard form for any straight line, whether or not it goes through the origin.

3.2. The turn τ is called the *clinant* of the line, and it gives the orientation* of the line in somewhat the same way as the *slope* gives the orientation of a line in the Cartesian system. It will be helpful to the reader later if he will make himself thoroughly familiar with the relation between the clinant τ of a line and the orientation of the line. The simplest way to see this relationship is to picture it graphically. The following figure shows for certain values of the turn τ segments of lines having the clinant τ. It may be noted that as the turn τ runs

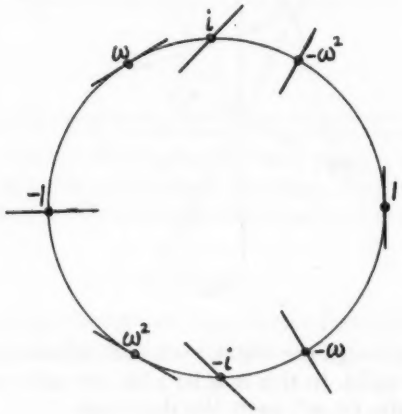

FIG. 8

completely around the unit circle counter-clockwise, the line segment with clinant τ swings around clockwise through 180°.

We now express this relationship analytically. Let ϕ be an angle through which the axis of reals may be turned to bring it into parallelism with the line $\tau x + y = \tau r$. There are infinitely many such angles ϕ, but always just one, ϕ_0 such that $0 \leqq \phi_0 < \pi$; and all other values of ϕ differ from ϕ_0 by multiples of π.

* We use the word *orientation* rather than *direction*, preferring to reserve the latter word for the thing indicated by a pointed finger or an arrowhead on a line. Thus "north" indicates a direction and "south" a different direction. A line, or line segment, without an arrowhead on it has orientation but not direction; and two parallel lines or segments have the same orientation but may have different directions.

This angle ϕ_0 is often called the *inclination* of the line. Since

$$\frac{1}{\tau} = e^{2i\theta},$$

we have

$$2i\theta = \log\left(\frac{1}{\tau}\right) = -\log(\tau),$$

and

$$\theta = -\frac{1}{2i}\log(\tau).$$

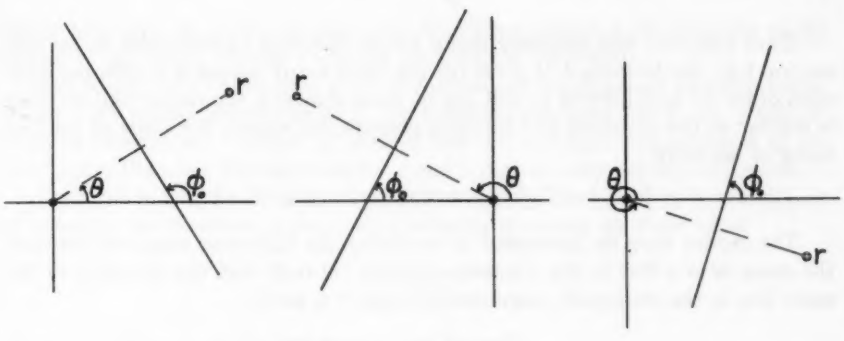

FIG. 9

Now when

$$0 \leqq \theta < \frac{\pi}{2}, \quad \phi_0 = \theta + \frac{\pi}{2};$$

when

$$\frac{\pi}{2} \leqq \theta < \frac{3\pi}{2}, \quad \phi_0 = \theta - \frac{\pi}{2} \quad \text{or} \quad \phi_0 + \pi = \theta + \frac{\pi}{2};$$

when

$$\frac{3\pi}{2} \leqq \theta < 2\pi, \quad \phi_0 = \theta - \frac{3\pi}{2} \quad \text{or} \quad \phi_0 + 2\pi = \theta + \frac{\pi}{2};$$

and hence there is always a value of ϕ such that

$$\phi = \theta + \frac{\pi}{2}.$$

Then

$$\phi = \frac{\pi}{2} - \frac{1}{2i}\log(\tau)$$

$$= -\frac{1}{2i}[-\pi i + \log(\tau)]$$

$$= \frac{i}{2}[\log(-1) + \log(\tau)]$$

and hence

$$\phi = \frac{i}{2}\log(-\tau). \tag{3.21}$$

Since $\log(-\tau)$ has infinitely many values differing by multiples of $2\pi i$ (see section 1.6), the formula 3.21 gives the infinitely many values of ϕ differing from each other by multiples of π, and one of these values is the inclination ϕ_0. This is similar to the situation in Cartesian coordinates where, the slope of the line being m, we have

$$\phi = \tan^{-1} m.$$

The reader may be interested in verifying the following relations between the slope m of a line in the Cartesian system (m real) and the clinant τ of the same line in the conjugate coordinate system (τ a turn):

$$m = \frac{i(\tau+1)}{\tau-1}, \qquad \tau = \frac{m+i}{m-i},$$
$$\tan^{-1} m = \frac{i}{2}\log(-\tau). \tag{3.22}$$

3.3 By the directed angle ϕ_{12} *from* a line l_1 *to* a line l_2 we shall mean any angle through which the line l_1 may be turned to bring it into parallelism with the line l_2. There are infinitely many values of ϕ_{12} differing from each other by multiples of π. From 3.21 we see that the angle *from* a line with clinant τ_1 *to* a line with clinant τ_2 is

$$\phi_{12} = \frac{i}{2}\log(-\tau_2) - \frac{i}{2}\log(-\tau_1)$$

or

$$\phi_{12} = \frac{i}{2}\log\left(\frac{\tau_2}{\tau_1}\right). \tag{3.31}$$

Evidently two lines are parallel if $\tau_1 = \tau_2$. If the lines are perpendicular we have

$$\phi_{12} = \frac{\pi}{2} + \pi s, \qquad s \text{ any integer},$$

$$\frac{i}{2} \log\left(\frac{\tau_2}{\tau_1}\right) = \frac{\pi}{2} + \pi s,$$

$$\log\left(\frac{\tau_2}{\tau_1}\right) = -\pi i - 2\pi s i,$$

$$\frac{\tau_2}{\tau_1} = e^{i(-\pi - 2\pi s)} = -1,$$

$$\tau_2 = -\tau_1.$$

Hence two lines are perpendicular when the clinant of one is the negative of the clinant of the other. This is a relation we shall use frequently.

The reader may verify that if the self-conjugate equations of two non-parallel lines are solved simultaneously, the resulting solutions for x and y will always be conjugates. If either or both of two self-conjugate equations is of degree higher than one the solutions for x and y may or may not be conjugates.

If τ_1 and τ_2 are the clinants of two intersecting lines, and τ_3 is the clinant of either of the bisectors of the angles between the lines, we must have

$$\frac{\tau_1}{\tau_3} = \frac{\tau_3}{\tau_2}.$$

Hence the clinants of the two bisectors must be

$$\tau_3 = \pm\sqrt{\tau_1\tau_2}. \tag{3.32}$$

3.4. Because of the linear form of the equation of a line, $ax+by+c=0$, it follows at once that the line through the two points (x_1, y_1) and (x_2, y_2) has the equation

$$\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0, \tag{3.41}$$

or

$$(y_1 - y_2)x - (x_1 - x_2)y + x_1y_2 - x_2y_1 = 0,$$

or

$$-\frac{y_1 - y_2}{x_1 - x_2}x + y = \frac{x_1y_2 - x_2y_1}{x_1 - x_2}.$$

Hence we see that the clinant of the line through the points (x_1, y_1) and (x_2, y_2) is

$$\tau = -\frac{y_1 - y_2}{x_1 - x_2}. \tag{3.42}$$

Also it is obvious at once that the equation of the line having the clinant τ and passing through the point (x_1, y_1) is

$$\tau x + y = \tau x_1 + y_1. \tag{3.43}$$

3.5. Let $\tau x + y = \tau r$ be the equation of a given line, and (x_1, y_1), (x_2, y_2) two points satisfying the condition

$$\tau x_1 + \bar{y}_1 = \tau r. \tag{3.51}$$

This implies also the conjugate relation

$$\frac{y_1}{\tau} + x_2 = \frac{\bar{r}}{\tau} = r$$

or

$$\tau x_2 + \bar{y}_1 = \tau r. \tag{3.52}$$

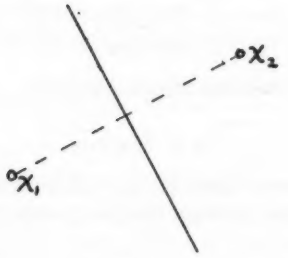

FIG. 10

Adding 3.51 and 3.52 and dividing by 2, we have

$$\tau\left(\frac{x_1 + x_2}{2}\right) + \left(\frac{y_1 + y_2}{2}\right) = \tau r,$$

which shows that the mid-point of the segment from x_1 to x_2 lies on the given line. If we subtract 3.52 from 3.51, we have

$$\tau(x_1 - x_2) + (y_2 - y_1) = 0$$

or

$$\tau = \frac{y_1 - y_2}{x_1 - x_2},$$

which shows that the given line is perpendicular to the segment from x_1 to x_2. It follows that each of the two points is the reflection of the other in the given line. We shall say that two such points are *inverse points with respect to the line*, and that each point is the inverse of the other with respect to the line.

If the equation of the line is in the form

$$ax + by + c = 0,$$

the condition that the points (x_1, y_1) and (x_2, y_2) be inverse points with respect to the line is

$$ax_1 + by_2 + c = 0, \tag{3.53}$$

or the equivalent condition

$$ax_2 + by_1 + c = 0.$$

If we use only the first coordinate to designate each point, it follows that the inverse of the point x_1 with respect to the line $ax+by+c=0$ (or $\tau x+y=\tau r$) is

$$x_2 = \frac{-b\bar{x}_1 - c}{a} = r - \frac{\bar{x}_1}{\tau}. \tag{3.54}$$

The distance between the point (x_1, y_1) and its reflection in the line $ax+by+c=0$ is

$$\sqrt{\left(\frac{-by_1 - c}{a} - x_1\right)\left(\frac{-ax_1 - c}{b} - y_1\right)} = \sqrt{\frac{(ax_1 + by_1 + c)^2}{ab}}$$

$$= \pm \frac{ax_1 + by_1 + c}{\sqrt{ab}}.$$

Hence the square of the distance from the point (x_1, y_1) to the line $ax+by+c=0$ is

$$d^2 = \frac{(ax_1 + by_1 + c)^2}{4ab}. \tag{3.55}$$

Chapter IV

THE CIRCLE

4.1. Let k be any complex number and g any complex number other than zero, and consider the circle whose center is the point k and whose radius r is

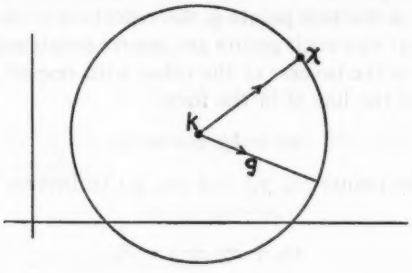

FIG. 11

the length of the vector g, so that $r^2 = g\bar{g}$. If x is any point on the circle, the vectors $x-k$ and g are of the same length and

$$x - k = gt,$$

where t is a turn varying with x. Hence

$$x = k + gt \tag{4.11}$$

is a map equation of the circle, the point x running around the circle as t runs through all turn values.

Equation 4.11 implies the conjugate equation

$$y = \bar{k} + \frac{\bar{g}}{t},$$

and, eliminating t, we obtain the self-conjugate equation of the circle

$$(x - k)(y - \bar{k}) = g\bar{g} = r^2. \tag{4.12}$$

Equation 4.12 is of the general form

$$axy + bx + cy + d = 0, \tag{4.13}$$

and we may ask under what conditions will an equation of this form represent a circle. Evidently we must have $a \neq 0$, for otherwise the equation would be linear and could only represent a line. Then in order that the equation be self-conjugate (see 2.23) it is necessary that

$$a = \lambda\bar{a}, \qquad b = \lambda\bar{c}, \qquad c = \lambda\bar{b}, \qquad d = \lambda\bar{d},$$

from which it follows that d/a must be real and b/a and c/a must be conjugates. Then the equation 4.13 may be written

$$\left(x+\frac{c}{a}\right)\left(y+\frac{b}{a}\right)=\frac{bc}{a^2}-\frac{d}{a}, \tag{4.14}$$

and $bc/a^2-d/a$ is real. Now comparing this with equation 4.12, we see that it represents a circle if $bc/a^2-d/a$ is positive. Hence, under the conditions

$$\begin{cases} a \neq 0, \\ d/a \text{ is real}, \\ b/a \text{ and } c/a \text{ are conjugates}, \\ (bc-ad)/a^2 > 0, \end{cases} \tag{4.15}$$

equation 4.13 represents a circle with center at $-c/a$ and radius $\sqrt{(bc-ad)/a^2}$.

If x_1, x_2, x_3 are any three points not on a line, the equation

$$\begin{vmatrix} xy & x & y & 1 \\ x_1y_1 & x_1 & y_1 & 1 \\ x_2y_2 & x_2 & y_2 & 1 \\ x_3y_3 & x_3 & y_3 & 1 \end{vmatrix} = 0 \tag{4.16}$$

is obviously satisfied by the coordinates of each of the three points, and is of the form 4.13 with coefficients satisfying conditions 4.15. It is therefore the equation of the circle through the three points.*

4.2. Two points are *inverse points with respect to a circle* (a proper circle, $r \neq 0$) if they lie on a line with the center, both on the same side of the center, and the product of their distances from the center is equal to the square of the radius. From this definition certain geometric facts are immediately obvious:

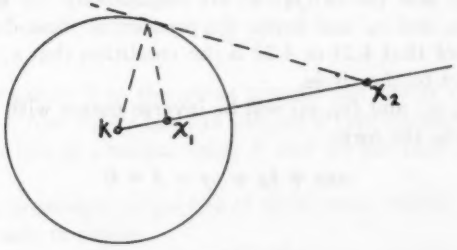

FIG. 12

if one of the points is inside the circle the other is outside; if one of the points is given, anywhere in the plane except at the center of the circle, the other is uniquely determined and may be easily constructed; and a point coincides with its inverse point when and only when it is on the circle.

* If the three points are on a line the equation is linear (the coefficient of xy vanishes) and is the equation of the line through the three points.

We now show that two points (x_1, y_1) and (x_2, y_2) are inverse points with respect to the circle

$$(x - k)(y - \bar{k}) = r^2$$

if and only if

$$(x_1 - k)(y_2 - \bar{k}) = r^2. \tag{4.21}$$

Condition 4.21 implies the conjugate condition

$$(x_2 - k)(y_1 - \bar{k}) = r^2. \tag{4.22}$$

From 4.21

$$x_1 - k = \frac{r^2}{y_2 - \bar{k}}$$

or

$$x_1 - k = \frac{r^2}{(y_2 - \bar{k})(x_2 - k)}(x_2 - k).$$

Since

$$\frac{r^2}{(y_2 - \bar{k})(x_2 - k)}$$

is real and positive, it follows that the points x_1 and x_2 are on a line with the center k and on the same side of k. Now multiplying 4.21 and 4.22 we have

$$(x_1 - k)(y_1 - \bar{k})(x_2 - k)(y_2 - \bar{k}) = r^4.$$

But $(x_1 - k)(y_1 - \bar{k})$ and $(x_2 - k)(y_2 - \bar{k})$ are respectively the squares of the distances from k to x_1 and x_2, and hence the product of these distances is r^2. This completes the proof that 4.21 or 4.22 is the condition that x_1 and x_2 be inverse points with respect to the circle.

The points (x_1, y_1) and (x_2, y_2) will be inverse points with respect to a circle whose equation is in the form

$$axy + bx + cy + d = 0$$

if and only if

$$ax_1y_2 + bx_1 + cy_2 + d = 0, \tag{4.23}$$

or (an equivalent condition) if and only if

$$ax_2y_1 + bx_2 + cy_1 + d = 0. \tag{4.24}$$

Using only the first coordinate to designate each point, the inverse of the point x_1 with respect to the circle

$$axy + bx + cy + d = 0$$

is

$$x_2 = \frac{-cx_1 - d}{ax_1 + b}. \tag{4.25}$$

Note that if we drop the restriction $a \neq 0$, the equation

$$axy + bx + cy + d = 0$$

includes both straight lines and circles; and when $a = 0$, 4.23 and 4.25 above reduce to 3.53 and 3.54 giving inverse points with respect to a line.

4.3. Let P_1 and P_2 be inverse points with respect to any circle with center C. Then the line l_1 through P_2 and perpendicular to the line CP_1P_2 is called the *polar line of the point P_1 with respect to the circle;* and similarly the line l_2 through P_1 and perpendicular to CP_1P_2 is the polar line of P_2. Thus, with respect to a

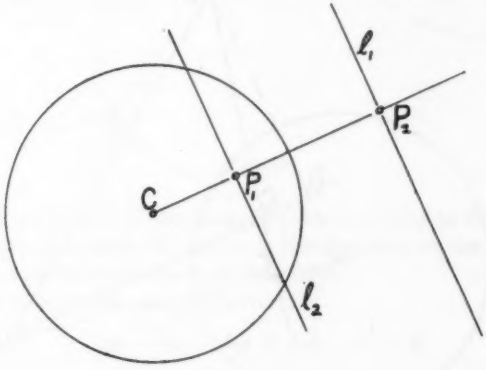

FIG. 13

fixed circle, every point P in the plane (except the center of the circle) has a unique polar line l. Also every line l in the plane (not through the center of the circle) is the polar line of a unique point P, and we say that P is the *polar point of the line l.*

The following geometric properties of these polar points and lines are quite obvious or very easy to prove.

(a) The polar line of P cuts the circle or not according as P is outside or inside the circle. If P is on the circle, the polar line is the tangent to the circle at P.

(b) The polar line of P passes through P if and only if P lies on the circle.

(c) If the line l cuts the circle at A and B, the tangents to the circle at these points intersect at the polar point of l.

(d) If the point B lies on the polar line of A, then A lies on the polar line of B.

The reader may show that the polar line of the point a with respect to the circle $(x-k)(y-\bar{k})=r^2$ has the self-conjugate equation

$$(\bar{a} - \bar{k})x + (a - k)y = k\bar{a} + \bar{k}a - 2k\bar{k} + 2r^2,$$

and that the equation of the tangent to the circle whose map equation is $x=k+gt$ at the point given by $t=t_1$ is

$$\bar{g}x + gt_1^2 y = k\bar{g} + \bar{k}gt_1^2 + 2g\bar{g}t_1.$$

With respect to a given circle a triangle is said to be *self-polar* (or *self-conjugate*) if each side of the triangle is the polar line of the opposite vertex. Each altitude of such a self-polar triangle must evidently pass through the center of the circle, and hence the center of the circle is the orthocenter of any self-polar

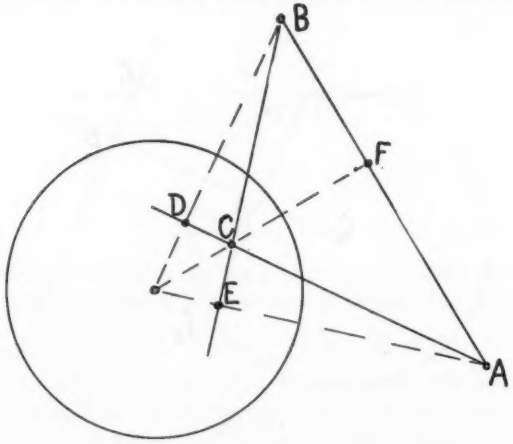

FIG. 14

triangle. Moreover, if the altitudes from the vertices A, B, and C respectively meet the opposite sides at the points D, E, and F, then A, D and B, E and C, F are pairs of inverse points with respect to the circle.

Given the circle, a self-polar triangle may be easily constructed. One vertex may be chosen arbitrarily, and a second vertex anywhere on the polar line of the first vertex; the third vertex is then uniquely determined.

Given a triangle, is there a circle with respect to which the triangle is self-polar? This question will be discussed in a later chapter (see section 7.7).

4.4. Suppose the equation of any circle is in the form

$$(x - k)(y - \bar{k}) - r^2 = 0 \quad \text{or} \quad xy + bx + cy + d = 0,$$

the coefficient of the xy term being 1. Then the value of the left member of

the equation when we substitute for x and y the coordinates (x_1, y_1) of any point in the plane is called the *power of the point* (x_1, y_1) *with respect to the circle.* Since $(x_1-h)(y_1-\bar{h})$ is the square of the distance of the point from the center of the circle, we see that when the point is outside the circle the power is positive and

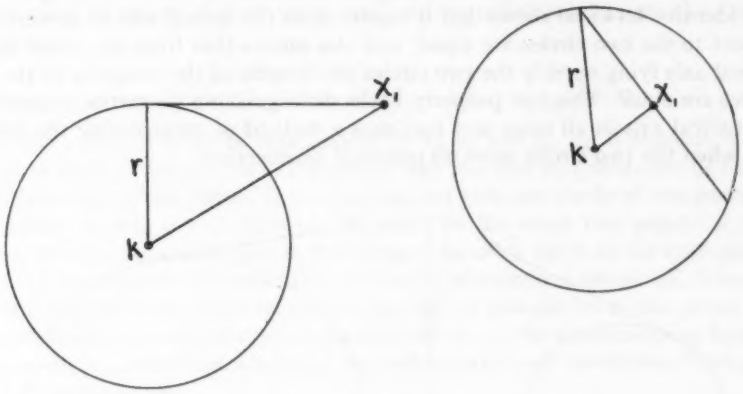

FIG. 15

is the square of the length of the tangent from the point to the circle, and when the point is inside the circle the power is the negative of the square of half the length of the minimum chord through the point.

Consider the two circles, not concentric,

$$xy - \bar{k}_1 x - k_1 y + k_1\bar{k}_1 - r_1^2 = 0 \tag{4.41}$$

and

$$xy - \bar{k}_2 x - k_2 y + k_2\bar{k}_2 - r_2^2 = 0, \qquad k_1 \neq k_2, \tag{4.42}$$

and the line

$$(\bar{k}_2 - \bar{k}_1)x + (k_2 - k_1)y + k_1\bar{k}_1 - k_2\bar{k}_2 + r_2^2 - r_1^2 = 0, \tag{4.43}$$

obtained by eliminating the xy term between the equations of the two circles. This line is called the *radical axis* of the two circles, and we shall show that there is a simple geometric relationship between this radical axis and the two circles.

Since we have the identity (in x and y)

$$\begin{aligned}(xy - \bar{k}_1 x - k_1 y + k_1\bar{k}_1 - r_1^2) - (xy - \bar{k}_2 x - k_2 y + k_2\bar{k}_2 - r_2^2) \\ \equiv \{(\bar{k}_2 - \bar{k}_1)x + (k_2 - k_1)y + k_1\bar{k}_1 - k_2\bar{k}_2 + r_2^2 - r_1^2\},\end{aligned} \tag{4.44}$$

we see that if the two circles intersect in two points the radical axis will go through these two points, and if the radical axis cuts one of the circles in two

points it must cut the other circle in the same two points. The clinant of the radical axis is $(\bar{k}_2-\bar{k}_1)/(k_2-k_1)$, and this is the negative of the clinant of the line through the centers k_1 and k_2 (see 3.42), which shows that the radical axis of two circles is perpendicular to the line of centers. It follows that if the two circles are tangent to each other, the radical axis is the common tangent line. The identity 4.44 also shows that if a point is on the radical axis its powers with respect to the two circles are equal; and this means that from any point of the radical axis lying outside the two circles the lengths of the tangents to the two circles are equal. This last property is the distinguishing geometric property of the radical axis in all cases and furnishes a method of constructing the radical axis when the two circles have no points of intersection.

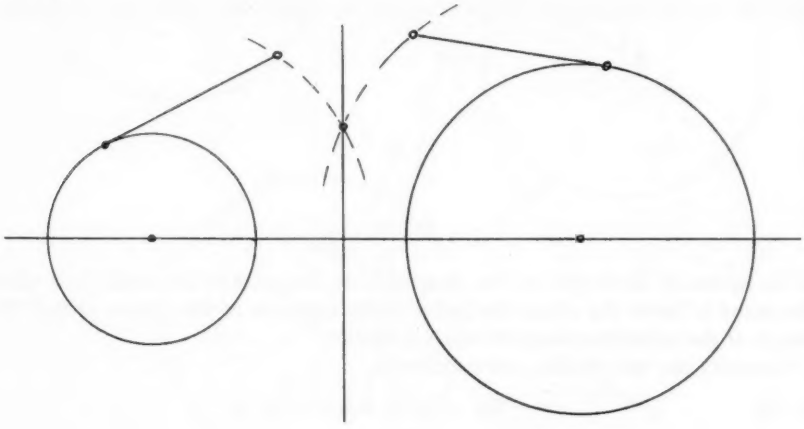

FIG. 16

4.5. Consider again the two circles 4.41 and 4.42 of the preceding section. If we multiply 4.41 by λ_1 and 4.42 by λ_2 and then add, λ_1 and λ_2 being *real* constants such that $\lambda_1+\lambda_2\neq 0$*, we obtain the equation of a third circle

$$(\lambda_1+\lambda_2)xy-(\lambda_1\bar{k}_1+\lambda_2\bar{k}_2)x-(\lambda_1 k_1+\lambda_2 k_2)y+\lambda_1 k_1\bar{k}_1+\lambda_2 k_2\bar{k}_2-\lambda_1 r_1^2-\lambda_2 r_2^2=0. \tag{4.51}$$

For varying values of λ_1 and λ_2 this equation represents an infinite set of circles which we call a *pencil of circles* (also sometimes called a coaxial set of circles); and the geometric property of such a pencil of circles is that any two circles of the set have the same radical axis as any other two circles of the set. To show this let the circle 4.51 for some fixed λ_1 and λ_2, and the circle

* We take λ_1 and λ_2 to be real in order to make equation 4.51 self-conjugate, and the restriction $\lambda_1+\lambda_2\neq 0$ is to insure that equation 4.51 shall represent a circle rather than a line.

$$(4.52)\qquad (\mu_1+\mu_2)xy-(\mu_1\bar{k}_1+\mu_2\bar{k}_2)x-(\mu_1k_1+\mu_2k_2)y + \mu_1k_1\bar{k}_1+\mu_2k_2\bar{k}_2-\mu_1r_1^2-\mu_2r_2^2=0$$

be any two different circles of the pencil (we must have $\lambda_1\mu_2-\lambda_2\mu_1\neq 0$ in order that these circles should not be the same). We find the radical axis of these two circles by eliminating the xy term, and obtain, after taking out the factor $\lambda_1\mu_2-\lambda_2\mu_1$,

$$(4.53)\qquad (\bar{k}_2-\bar{k}_1)x+(k_2-k_1)y+k_1\bar{k}_1-k_2\bar{k}_2+r_2^2-r_1^2=0.$$

Since this equation is quite independent of λ_1, λ_2 and μ_1, μ_2, it represents the radical axis of any two circles of the pencil; and this line may therefore be called the *radical axis of the pencil.* If the radical axis cuts one circle of the pencil in two points, it cuts every circle of the pencil in the same two points; if it is tangent to one circle of the pencil, it is tangent to every circle at the same point; and if it has no points of intersection with some one circle of the pencil, it has no intersections with any circle of the pencil, and no two circles of the pencil cut each other. There are three types of pencil as shown in the accompanying figures. In all cases, tangents from a point of the radical axis to all the circles of the pencil are of equal length.

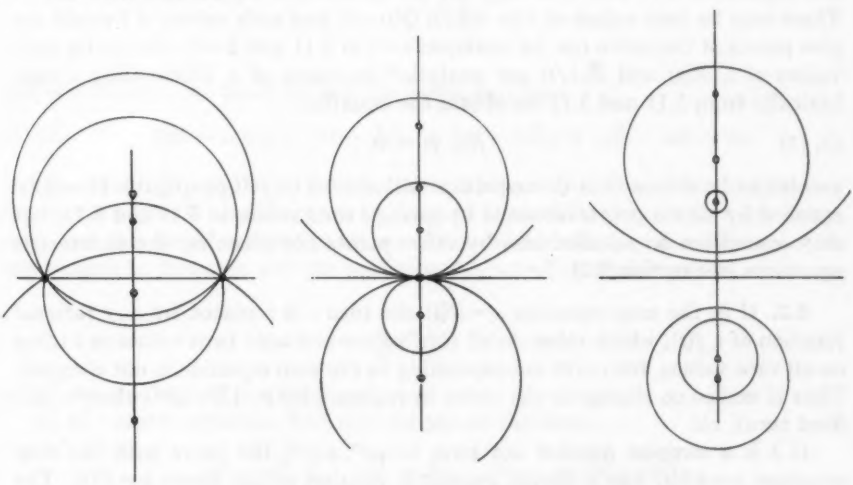

FIG. 17

Chapter V

MAP EQUATIONS AND THE DERIVATIVE $D_t x$

5.1. The map equations 3.11, 3.14, and 4.11 of the straight line and circle are examples of a kind of representation which can be used advantageously in the study of certain other rational (unicursal) curves.

Suppose the complex variable x is equated to a rational function of a variable turn t,

$$x = R(t) = \frac{P(t)}{Q(t)}, \tag{5.11}$$

where $P(t)$ and $Q(t)$ are relatively prime polynomials. As the point t runs around the unit circle the point x describes a rational curve. Equation 5.11 implies the conjugate equation

$$y = \overline{R}\left(\frac{1}{t}\right), \tag{5.12}$$

and equations 5.11 and 5.12 together are parametric equations of the curve. There may be turn values of t for which $Q(t)=0$, and such values of t would not give points of the curve (as, for example, $t=1$ in 3.11 and 3.14). Except for such values of t, $R(t)$ and $\overline{R}(1/t)$ are analytic* functions of t. Eliminating t algebraically from 5.11 and 5.12 we obtain the equation

$$f(x, y) = 0, \tag{5.13}$$

and it can be shown that this equation will always be self-conjugate. It will be satisfied by all the points obtained by giving t turn values in 5.11 and 5.12, but may sometimes be satisfied also by other points not given by the parametric equations (see section 9.2).

5.2. If in the map equation $x=R(t)$ the turn t is replaced by any rational function of t, $f(t)$, which takes on all turn values and only turn values as t takes on all turn values, the curve corresponding to the map equation is not changed. Thus it makes no change in the curve to replace t by t^n, $1/t^n$, $t_1 t^n$ (where t_1 is a fixed turn), *etc.*

If b is a complex number not zero, $b=\rho e^{i\theta}$, $\rho \neq 0$, the curve with the map equation $x=bR(t)$ has a simple geometric relation to the curve $x=R(t)$. The geometric effect of the multiplication by b is to rotate the curve through an angle θ about the origin, and to expand (or contract) it about the origin to ρ times its original size. If b is a turn, $\rho=1$, the curve is merely rotated without change of size or shape. If a complex number a is added to $R(t)$, the effect is to translate the curve without change of size or shape, the direction and distance

* See Curtiss, Analytic Functions of a Complex Variable, pp. 43–51.

of the translation being the direction and length of the vector a. Thus the curve

$$x = a + bR(t) \tag{5.21}$$

is similar to the curve $x=R(t)$, the one being obtained from the other by the operations of rotation, expansion, and translation. If any rigid motion is applied to the curve $x=R(t)$, a map equation of the curve in the new position will be

$$x = a + t_1 R(t), \tag{5.22}$$

where t_1 is a turn.

5.3. The map equations 3.11 and 3.14 for a line and 4.11 for a circle are all included in the general form

$$x = \frac{a+bt}{c+dt}, \qquad \begin{vmatrix} a & b \\ c & d \end{vmatrix} \neq 0, \tag{5.31}$$

and we now show that this more general map equation always represents either a line or a circle.

Equation 5.31 implies the conjugate equation

$$y = \frac{\bar{a}t + \bar{b}}{\bar{c}t + \bar{d}},$$

and eliminating t we have the self-conjugate equation

$$(d\bar{d} - c\bar{c})xy + (\bar{a}c - \bar{b}d)x + (a\bar{c} - b\bar{d})y + (b\bar{b} - a\bar{a}) = 0. \tag{5.32}$$

If $d\bar{d}-c\bar{c}=0$, this equation is linear and satisfies conditions 2.31, and hence represents a line. With $d\bar{d}-c\bar{c}=0$, both c and d would be zero if either one were zero, and this would contradict the condition $ad-bc\neq 0$. Hence *neither* c nor d can be zero in this case and the condition $d\bar{d}-c\bar{c}=0$ is equivalent to

$$\frac{c}{d} = \frac{\bar{d}}{\bar{c}},$$

which means that c/d is a turn.

If $d\bar{d}-c\bar{c}\neq 0$, equation 5.31 may be put in the form

$$\left(x - \frac{b\bar{d} - a\bar{c}}{d\bar{d} - c\bar{c}}\right)\left(y - \frac{\bar{b}d - \bar{a}c}{d\bar{d} - c\bar{c}}\right) = \frac{(bc - ad)(\bar{b}\bar{c} - \bar{a}\bar{d})}{(d\bar{d} - c\bar{c})^2};$$

and since $bc-ad\neq 0$, $(bc-ad)(\bar{b}\bar{c}-\bar{a}\bar{d})>0$, and we have the equation of a circle with center at

$$\left(\frac{b\bar{d} - a\bar{c}}{d\bar{d} - c\bar{c}}, \frac{\bar{b}d - \bar{a}c}{d\bar{d} - c\bar{c}}\right)$$

and radius

$$r = \sqrt{\frac{(bc - a\bar{d})(\bar{b}c - \bar{a}\bar{d})}{(d\bar{d} - c\bar{c})^2}}.$$

We have then shown that the map equation

$$x = \frac{a + bt}{c + dt}, \quad \text{with} \quad \begin{vmatrix} a & b \\ c & d \end{vmatrix} \neq 0,$$

represents a line or a circle according as c/d is or is not a turn.

5.4. Suppose that the map equation of a curve is $x = R(t) = P(t)/Q(t)$ and that t_1 and t_2 are two distinct values of the turn t, with $Q(t_1) \neq 0$, and hence $Q(t_2) \neq 0$ when t_2 is sufficiently close to t_1; and let (x_1, y_1) and (x_2, y_2) be the two corresponding points on the curve. The clinant of the secant line through these two points is (see 3.42)

$$\tau = -\frac{y_1 - y_2}{x_1 - x_2} = -\frac{\dfrac{y_1 - y_2}{t_1 - t_2}}{\dfrac{x_1 - x_2}{t_1 - t_2}}.$$

If now t_1 is fixed and t_2 is made to approach t_1 *taking on only turn values*, the clinant of the secant will approach the clinant of the tangent to the curve at the point (x_1, y_1), and the fractions $(y_1 - y_2)/(t_1 - t_2)$ and $(x_1 - x_2)/(t_1 - t_2)$ will approach limits, the values of the formal derivatives* $D_t y$ and $D_t x$ for $t = t_1$. Thus, provided $D_t x \neq 0$ when $t = t_1$, the clinant of the tangent to the curve at the point (x_1, y_1) is

$$\tau = -\left.\frac{D_t y}{D_t x}\right]_{t=t_1}. \tag{5.41}$$

If either $D_t y$ or $D_t x$ is zero for $t = t_1$, the other is also. For with $x = R(t)$ we may write $D_t x = R'(t)$. Then $y = \overline{R}(1/t)$ and

$$D_t y = \overline{R}'\left(\frac{1}{t}\right)\left(-\frac{1}{t^2}\right) = \overline{R'}\left(\frac{1}{t}\right)\left(-\frac{1}{t^2}\right),$$

and since $-1/t^2$ cannot vanish, $D_t y$ vanishes when and only when $\overline{R}'(1/t) = 0$. But $\overline{R}'(1/t)$ is the conjugate of $R'(t) = D_t x$; and hence $D_t y$ vanishes when and only when $D_t x$ vanishes. This will occur only at certain singular points of the

* By the "formal derivatives" $D_t y$ and $D_t x$ we mean the functions of t obtained by applying to the functions $\overline{R}(1/t)$ and $R(t)$ the formulas used in differentiating rational functions of a *real* variable.

curve (as we shall see later), and at such points the formula 5.41 is meaningless. Because of the relation

$$D_t y = -\overline{D_t x}/t^2,$$

formula 5.41 may be written

$$\tau = \frac{\overline{D_t x}}{t^2 D_t x}\Bigg]_{t=t_1}, \tag{5.42}$$

a useful form which enables us to obtain the clinant of the tangent from the map equation 5.11 without writing the conjugate equation 5.12.

5.5. Formula 5.41 and the obvious analogy with slope in the Cartesian system suggest that the clinant of the tangent to the curve might be given by the negative of the formal derivative $D_x y$ obtained from the self-conjugate equation 5.13 by either solving for y in terms of x or using the implicit function method. This procedure would (with slight restrictions) be sound, but it will be simpler for our purposes to use 5.42. The reader may, however, be interested in checking the results against each other in the simple case of the circle, finding the clinant of the tangent by (1) using 5.42 with the map equation 4.11, (2) solving the self-conjugate equation 4.14 for y in terms of x and using $-D_x y$ as a function of x, and (3) using $-D_x y$ as a function of x and y obtained from 4.14 by the implicit function method.

Chapter VI

SYMMETRIC FUNCTIONS OF TURNS

6.1. Let n turns, $t_1, t_2, \cdots, t_n$, be the roots of the equation

$$t^n - s_1 t^{n-1} + s_2 t^{n-2} + \cdots + (-1)^{n-1} s_{n-1} t + (-1)^n s_n = 0. \tag{6.11}$$

Then the coefficient s_1 is the sum of the n turns, s_2 is the sum of the products of the turns taken two at a time, and in general s_i is the sum of the products of the turns taken i at a time. Since s_n is the product of all the turns, it is itself a turn, and hence

$$\mathfrak{s}_n = \frac{1}{s_n}. \tag{6.12}$$

Also we have

$$\mathfrak{s}_1 = \frac{1}{t_1} + \frac{1}{t_2} + \cdots + \frac{1}{t_n} = \frac{s_{n-1}}{s_n},$$

and in general

$$\mathfrak{s}_i = \frac{s_{n-i}}{s_n}, \qquad i = 1, 2, \cdots, n-1. \tag{6.13}$$

These s_i are called the *elementary symmetric functions** of the turns, and any rational symmetric function of the turns can be expressed as a rational function of the s_i. Thus

$$(t_2 + t_3)(t_3 + t_1)(t_1 + t_2) = s_1 s_2 - s_3. \tag{6.14}$$

Since the turns t_i are the roots of the equation 6.11 we have *for each* t_i the identity

$$t_i^n - s_1 t_i^{n-1} + s_2 t_i^{n-2} \cdots + (-1)^n s_n = 0, \tag{6.15}$$

and this identity will be used frequently in the later chapters. Note also for future use that for any three t_i, $s_1 s_2 / s_3$ and $s_3/(s_1 s_2 - s_3)$ are real; and for three distinct t_i, $s_1 s_2 - 9 s_3 \neq 0$.

6.2. Let $\sum_1, \sum_2, \sum_3, \sum_4$ be the elementary symmetric functions for four turns t_1, t_2, t_3, t_4, and s_1, s_2, s_3 the elementary symmetric functions for three of the four turns when one of them, say t_k, is omitted. It is readily verified that

* Bôcher, Higher Algebra, p. 243. Dickson, New First Course in the Theory of Equations, pp. 17–18 and p. 177.

$$(6.21)\qquad \begin{cases} \Sigma_1 = s_1 + t_k \\ \Sigma_2 = s_2 + s_1 t_k \\ \Sigma_3 = s_3 + s_2 t_k \\ \Sigma_4 = \quad s_3 t_k. \end{cases}$$

To express the s_i in terms of the $\sum_i$ and t_k we may obtain the expression for s_1 from the first of the equations 6.21 and then s_2 and s_3 successively from the second and third equations; or we may obtain s_3 from the fourth equation and then s_2 and s_1 from the third and second equations. The results are

$$(6.22)\qquad \begin{cases} s_1 = \Sigma_1 - t_k & = \dfrac{\Sigma_2 t_k^2 - \Sigma_3 t_k + \Sigma_4}{t_k^3} \\ s_2 = \Sigma_2 - \Sigma_1 t_k + t_k^2 & = \dfrac{\Sigma_3 t_k - \Sigma_4}{t_k^2} \\ s_3 = \Sigma_3 - \Sigma_2 t_k + \Sigma_1 t_k^2 - t_k^3 = \dfrac{\Sigma_4}{t_k}. \end{cases}$$

That the two different expressions for each s_i are equal may be seen at once from the identity 6.15.

More generally, if $\sum_1, \sum_2, \cdots, \sum_n$ are the symmetric functions for n turns, and $s_1, s_2, \cdots, s_{n-1}$ for the $n-1$ turns when t_k is omitted, then

$$\begin{aligned} \Sigma_1 &= s_1 + t_k \\ \Sigma_i &= s_i + s_{i-1} t_k, \qquad i = 2, 3, \cdots, n-1, \\ \Sigma_n &= \quad s_{n-1} t_k; \end{aligned}$$

and the s_i may be expressed in terms of the $\sum_i$ and t_k by expressions similar to those in 6.22.

6.3. If a rational function of n turns is symmetrical in $n-1$ of the turns (but not in all n) it may be expressed in many ways as a rational function of the odd turn and the elementary symmetric functions of the n turns. In the following chapters we shall make frequent use of this fact. For example,

$$t_2 t_3 (t_2 + t_1)(t_1 + t_3) = \frac{s_3(s_1 s_2 - s_3)}{t_1(s_1 - t_1)} = s_1 s_3 + \frac{s_3^2}{t_1^2}$$

and

$$\frac{t_2 t_3}{t_2 + t_3} + \frac{t_3 t_1}{t_3 + t_1} = \frac{s_3^2 - s_3^2/t_3^2}{s_1 s_2 - s_3}$$

where the s_i are for the *three* turns t_1, t_2, t_3.

Chapter VII

SOME GEOMETRY OF THE TRIANGLE

7.1. Given any triangle we may fix our coordinate system so that the circumcircle of the triangle will be the unit circle. Then the vertices of the triangle will correspond to three turns, t_1, t_2, t_3, which are roots of an equation

$$t^3 - s_1 t^2 + s_2 t - s_3 = 0,$$

and the circumcenter of the triangle will be the point 0. The centroid of the triangle (see section 1.7) will be the point $(t_1+t_2+t_3)/3$ or $s_1/3$.

To find the orthocenter of the triangle (the point of intersection of the three perpendiculars from the vertices to the opposite sides) we note that the clinant of the side t_2t_3 is (see 3.42)

$$-\frac{\frac{1}{t_2}-\frac{1}{t_3}}{t_2-t_3}=\frac{1}{t_2t_3},$$

and hence the equation of this side is (see 3.43)

$$\frac{x}{t_2t_3}+y=\frac{1}{t_2}+\frac{1}{t_3}. \tag{7.11}$$

Then the equation of the perpendicular from the vertex t_1 to the side t_2t_3 is

$$-\frac{x}{t_2t_3}+y=-\frac{t_1}{t_2t_3}+\frac{1}{t_1}. \tag{7.12}$$

Similarly the equation of the perpendicular from t_2 to the side t_1t_3 is

$$-\frac{x}{t_1t_3}+y=-\frac{t_2}{t_1t_3}+\frac{1}{t_2}.$$

Eliminating y between these two equations and solving for x we have $x=t_1+t_2+t_3$, or

$$x = s_1, \tag{7.13}$$

and hence the orthocenter of the triangle is the point s_1. Although we found this point as the intersection of only two of the perpendiculars, the symmetry of the result shows at once that the third perpendicular also goes through it.

If the triangle is equilateral we have (see 1.73 and 1.74)

either $t_2 = \omega t_1$, $t_3 = \omega^2 t_1$ and hence $t_1 + \omega t_2 + \omega^2 t_3 = 0$,

or $t_2 = \omega^2 t_1$, $t_3 = \omega t_1$ and hence $t_1 + \omega^2 t_2 + \omega t_3 = 0$.

In either case $s_1=0$, so that the circumcenter, centroid, and orthocenter coincide. Also $\bar{s}_1=s_2/s_3=0$, and therefore $s_2=0$.

If the triangle is not equilateral, the circumcenter, centroid, and orthocenter are respectively the distinct points, 0, $s_1/3$, and s_1; and these points obviously lie on a line (see section 1.4) with the centroid between the other two and at one-third the distance from the circumcenter to the orthocenter. This line is known as the *Euler line* of the triangle. Its clinant is

$$-\frac{\bar{s}_1}{s_1}=-\frac{s_2}{s_1 s_3},$$

and its equation is

$$s_3x-s_1s_3y=0. \tag{7.14}$$

The reader will find the following theorem very easy to prove:

If P_i, $i=1, 2, 3, 4$, are four points on a circle, and H_i is the orthocenter of the triangle whose vertices are the remaining three points when P_i is omitted, then the four line segments P_iH_i have a common mid-point, and hence the figures formed by the points P_i and H_i are congruent.

7.2. The mid-point of the side t_2t_3 of the triangle is $\frac{1}{2}(t_2+t_3)$ or $\frac{1}{2}(s_1-t_1)$. If now we write the map equation

$$x=\frac{s_1}{2}-\frac{1}{2}t, \tag{7.21}$$

it corresponds (see 4.11) to the circle with center at $s_1/2$ and radius $\frac{1}{2}$. Moreover, as the variable turn t takes the values t_1, t_2, t_3, the circle goes through the mid-points of the three sides. This circle, called the *nine-point circle* for reasons which will appear shortly, is half as large as the circumcircle (whose radius was taken as the unit length) and has its center $s_1/2$, the *nine-point center*, on the Euler line midway between the circumcenter and orthocenter.

We next show that this nine-point circle (Figure 18) also passes through the three feet of the altitudes from each vertex to the opposite side. The foot of the altitude from the vertex t_1 to the side t_2t_3 is found by solving the equations 7.11 and 7.12 simultaneously. Eliminating y between these two equations and solving for x, we have

$$x=\frac{1}{2}\left(t_1+t_2+t_3-\frac{t_2t_3}{t_1}\right),$$

and hence the foot of the altitude from t_1 to the side t_2t_3 is the point

$$x=\frac{s_1}{2}-\frac{s_3}{2t_1^2}. \tag{7.22}$$

Now we write the map equation

$$x = \frac{s_1}{2} - \frac{s_3}{2}t, \tag{7.23}$$

which corresponds to the circle with center at $s_1/2$ and radius $\left|-s_3/2\right| = \frac{1}{2}$. This is then the same nine-point circle, and as t takes the *turn* values $1/t_1^2$, $1/t_2^2$, and $1/t_3^2$ it evidently goes through the three feet of the altitudes from t_1, t_2, and t_3 respectively.

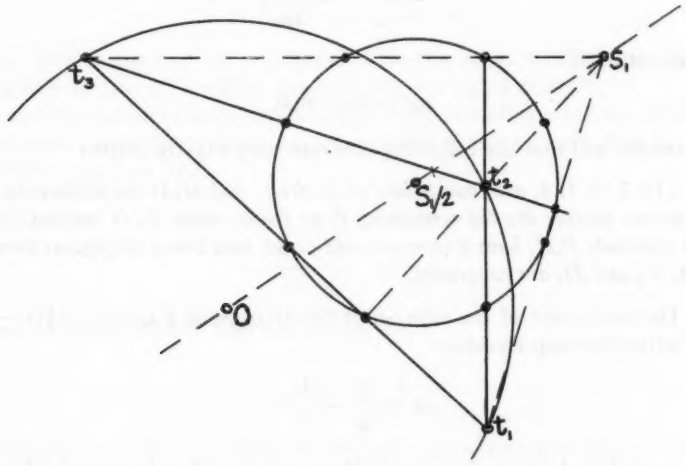

FIG. 18

We leave as an exercise for the reader to prove, *using the same method we have used twice above*, that the nine-point circle also goes through the three points mid-way between the orthocenter and each vertex of the triangle. We see then that this circle goes through the nine points which account for its name. It is obvious that when the triangle is isosceles, equilateral, or a right triangle, certain of these points will coincide.

Eliminating the parameter t between the map equation 7.21 and the conjugate equation

$$y = \frac{s_2}{2s_3} - \frac{1}{2t},$$

we have

$$\left(x - \frac{s_1}{2}\right)\left(y - \frac{s_2}{2s_3}\right) = \frac{1}{4} \tag{7.24}$$

as the self-conjugate equation of the nine-point circle.

It is easily seen geometrically that the four points consisting of the vertices of a triangle (not a right triangle) and its orthocenter t_1, t_2, t_3, s_1, make up an orthocentric set of points with the symmetric properties that each one of the four points is the orthocenter of the triangle whose vertices are the other three, that the line through any two of the points is perpendicular to the line through the other two, and that the nine-point circle for the triangle $t_1t_2t_3$ is also the nine-point circle for the triangles $s_1t_2t_3$, $t_1s_1t_3$, and $t_1t_2s_1$, and it may therefore be called the nine-point circle *for the orthocentric set.*

The proofs of the two following theorems will be good exercises for the reader.

Let P_i, $i=1, 2, 3, 4$, be an orthocentric set of points, C_i the circle through the three points other than P_i, and O_i the center of the circle C_i; then

(a) *the four circles C_i are all the same size;*

(b) *the four line segments P_iO_i have a common mid-point;*

(c) *the four points $P_iP_jO_kO_l$ ($i, j, k, l=1, 2, 3, 4$ in any order) are the vertices of a parallelogram.*

Let P_i, $i=1, 2, 3$, be the vertices of a triangle, D_i the foot of the altitude from P_i to the opposite side, and M_i the mid-point of this side; then the tangent to the circumcircle at P_i, the tangent to the nine-point circle at M_i, and the line D_jD_k, ($i, j, k=1, 2, 3$ in any order) are all parallel.

7.3 If a triangle with vertices at the points t_1, t_2, t_3 is a right triangle one of the turns must be the negative of another, that is, one of the quantities t_2+t_3, t_3+t_1, t_1+t_2 must be zero. But (see 6.14)

$$(t_2 + t_3)(t_3 + t_1)(t_1 + t_2) = s_1s_2 - s_3, \tag{7.31}$$

and therefore the condition for a right triangle is $s_1s_2-s_3=0$. If $t_2+t_3=0$, $s_1=t_1$, or (as is quite obvious) the orthocenter of a right triangle coincides with the vertex of the right angle.

In the case of an acute triangle (all three angles acute) the orthocenter lies inside the triangle and the distance from the circumcenter to the orthocenter is less than the radius of the circumcircle, that is, $|s_1|<1$ and $|s_1|^2<1$. But $|s_1|^2=s_1\bar{s}_1=s_1s_2/s_3$, and hence for an acute triangle $s_1s_2/s_3<1$ or $(s_1s_2-s_3)/s_3<0$.

Similarly for an obtuse triangle (one angle obtuse) we have $(s_1s_2-s_3)/s_3>0$.

Therefore triangle $t_1t_2t_3$ is

$$\begin{cases} \text{acute when} & (s_1s_2 - s_3)/s_3 < 0 \\ \text{right when} & (s_1s_2 - s_3)/s_3 = 0 \\ \text{obtuse when} & (s_1s_2 - s_3)/s_3 > 0. \end{cases} \tag{7.32}$$

7.4. Let us assume now that the triangle $t_1t_2t_3$ is not equilateral and is not a right triangle, so that $s_1\neq 0$, $s_2\neq 0$, and $s_1s_2-s_3\neq 0$. Then the tangents to the circumcircle at the vertices form a new triangle $P_1P_2P_3$, the vertex P_1 being the

intersection of the tangents at t_2 and t_3, *etc.* This triangle $P_1P_2P_3$ is called the *tangential triangle* of the triangle $t_1t_2t_3$.

Given any triangle $P_1P_2P_3$ there are four different circles each of which is tangent to the three lines P_2P_3, P_3P_1, P_1P_2. One of these circles lying inside the triangle is called the *incircle* (or inscribed circle) and the other three are called *excircles* (or escribed circles), and the centers of these circles are known as the *incenter* and the *excenters* of the triangle $P_1P_2P_3$. Starting, as in the preceding paragraph, with the triangle $t_1t_2t_3$, its circumcircle will be the incircle or one of the excircles of the tangential triangle $P_1P_2P_3$, depending on whether the triangle $t_1t_2t_3$ is acute or obtuse (in Figure 19 it is one of the excircles).

The tangent at t_2 has the reflex point $r_2=2t_2$, and its clinant is $\bar{r}_2/r_2=1/t_2^2$. Hence its equation is

$$\frac{x}{t_2^2}+y=\frac{2}{t_2}. \tag{7.41}$$

Similarly the tangent at t_3 has the equation

$$\frac{x}{t_3^2}+y=\frac{2}{t_3},$$

and solving these two equations for x we find that the point of intersection P_1 is

$$x=\frac{2t_2t_3}{t_2+t_3}.$$

This may be written

$$x=\frac{2t_2t_3(t_3+t_1)(t_1+t_2)}{(t_2+t_3)(t_3+t_1)(t_1+t_2)}=\frac{2\{t_1t_2t_3(t_1+t_2+t_3)+t_2^2t_3^2\}}{(t_2+t_3)(t_3+t_1)(t_1+t_2)}$$

$$=\frac{2\left(s_1s_3+\dfrac{s_3^2}{t_1^2}\right)}{s_1s_2-s_3}=\frac{2s_1s_3}{s_1s_2-s_3}+\frac{2s_3^2}{(s_1s_2-s_3)t_1^2}.$$

The three vertices P_i are then given by

$$x=\frac{2s_1s_3}{s_1s_2-s_3}+\frac{2s_3^2}{(s_1s_2-s_3)t_i^2}, \qquad i=1,2,3.$$

The map equation

$$x=\left(\frac{2s_3}{s_1s_2-s_3}\right)s_1+\left(\frac{2s_3}{s_1s_2-s_3}\right)t \tag{7.42}$$

represents a circle with

$$\text{center } \left(\frac{2s_3}{s_1s_2 - s_3}\right)s_1 \quad \text{and radius } \left|\frac{2s_3}{s_1s_2 - s_3}\right|,$$

and since it passes through the vertices P_1, P_2, and P_3 when t takes on the turn values s_3/t_1^2, s_3/t_2^2, and s_3/t_3^2, it follows that 7.42 is the map equation of the circumcircle of the tangential triangle $P_1P_2P_3$. Since $2s_3/(s_1s_2-s_3)$ is real (see section 6.1), the center of this circle lies on the line through O and s_1, and we have

THEOREM 7.43. *Given a triangle $t_1t_2t_3$ (not a right triangle or an equilateral triangle) and its tangential triangle $P_1P_2P_3$, the circumcenter of the triangle $P_1P_2P_3$ lies on the Euler line of the triangle $t_1t_2t_3$.*

The radius of the circumcircle of the triangle $P_1P_2P_3$ is

$$R = \left|\frac{2s_3}{s_1s_2 - s_3}\right| = \pm\frac{2s_3}{s_1s_2 - s_3},$$

the minus or plus sign to be used according as the triangle $t_1t_2t_3$ is acute or obtuse.

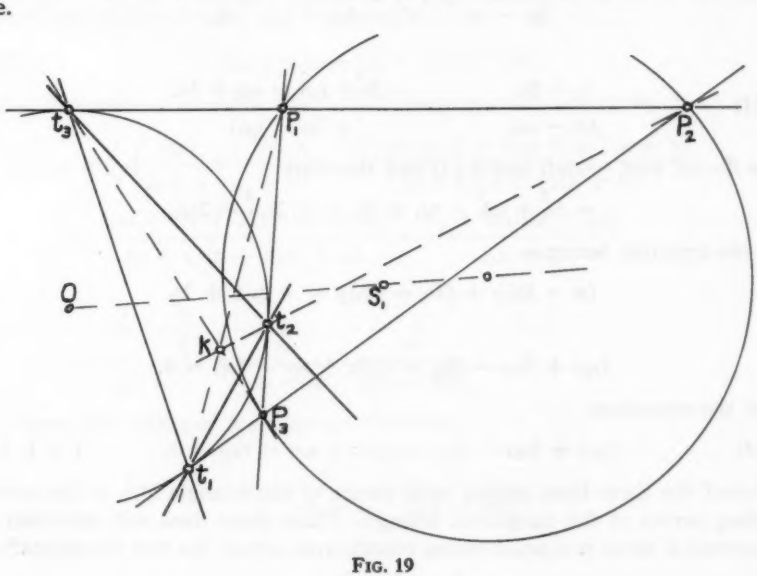

FIG. 19

The equation conjugate to 7.42 is

$$y = \frac{2s_2}{s_1s_2 - s_3} + \frac{2s_3}{(s_1s_2 - s_3)t},$$

and, eliminating t, we have

$$\left(x-\frac{2s_1s_3}{s_1s_2-s_3}\right)\left(y-\frac{2s_2}{s_1s_2-s_3}\right)=\frac{4s_3^2}{(s_1s_2-s_3)^2} \tag{7.44}$$

as the self-conjugate equation of the circumcircle of the tangential triangle $P_1P_2P_3$.

7.5. Consider the line through a vertex of a triangle and the corresponding vertex of the tangential triangle. The line through the points t_1 and $2t_2t_3/(t_2+t_3)$ has the clinant

$$-\frac{\dfrac{2}{t_2+t_3}-\dfrac{1}{t_1}}{\dfrac{2t_2t_3}{t_2+t_3}-t_1}=\frac{-2t_1+t_2+t_3}{t_1(2t_2t_3-t_3t_1-t_1t_2)}=\frac{s_1-3t_1}{3s_3-s_2t_1},$$

and hence its equation (see 3.43) is

$$\frac{s_1-3t_1}{3s_3-s_2t_1}x+y=\frac{s_1t_1-3t_1^2}{3s_3-s_2t_1}+\frac{1}{t_1}$$

or

$$\frac{s_1-3t_1}{3s_3-s_2t_1}x+y=\frac{-3t_1^3+s_1t_1^2-s_2t_1+3s_3}{t_1(3s_3-s_2t_1)}. \tag{7.51}$$

Now $t_1^3-s_1t_1^2+s_2t_1-s_3=0$ (see 6.15) and therefore

$$-3t_1^3+s_1t_1^2-s_2t_1+3s_3=-2s_1t_1^2+2s_2t_1,$$

and the equation becomes

$$(s_1-3t_1)x+(3s_3-s_2t_1)y=-2s_1t_1+2s_2$$

or

$$(s_1x+3s_3y-2s_2)-t_1(3x+s_2y-2s_1)=0.$$

Then the equations

$$(s_1x+3s_3y-2s_2)-t_i(3x+s_2y-2s_1)=0, \qquad i=1,2,3, \tag{7.52}$$

represent the three lines joining each vertex of the triangle $t_1t_2t_3$ to the corresponding vertex of the tangential triangle. These three lines will obviously be concurrent if there is a point whose coordinates satisfy the two equations*

$$s_1x+3s_3y-2s_2=0,$$
$$3x+s_2y-2s_1=0.$$

* These equations are not self-conjugate and hence do not represent lines, but they are simultaneously satisfied by the conjugate coordinates of a point.

Solving these equations, we find that they are satisfied by the pair of *conjugate* values

$$x = \frac{2s_2^2 - 6s_1s_3}{s_1s_2 - 9s_3}, \qquad y = \frac{2s_1^2 - 6s_2}{s_1s_2 - 9s_3}.$$

Note that $s_1s_2 - 9s_3 \neq 0$ (see section 6.1). We have then

THEOREM 7.53. *The three lines joining the vertices of the triangle $t_1t_2t_3$ to the corresponding vertices of the tangential triangle $P_1P_2P_3$ meet in a point k,*

$$k = \frac{2s_2^2 - 6s_1s_3}{s_1s_2 - 9s_3}. \tag{7.54}$$

This point is called the *symmedian point*† of the triangle $t_1t_2t_3$.

In the foregoing discussion the case of a right triangle, $s_1s_2 - s_3 = 0$, was excluded because in that case the tangential triangle does not exist. However, two of the three lines passing through the symmedian point k do exist, and hence k is determined for a right triangle also, and is given by formula 7.54.

The line through the circumcenter and the symmedian point is called the *Brocard line* of the triangle, and its equation is

$$(3s_2 - s_1^2)x + (s_2^2 - 3s_1s_3)y = 0. \tag{7.55}$$

7.6. The mid-point of the side P_1P_2 of the tangential triangle is

$$x = \frac{1}{2}\left(\frac{2t_2t_3}{t_2 + t_3} + \frac{2t_3t_1}{t_3 + t_1}\right) = \frac{t_3\{t_2(t_2 + t_1) + t_1(t_2 + t_3)\}(t_1 + t_2)}{(t_2 + t_3)(t_3 + t_1)(t_1 + t_2)}$$

$$= \frac{(t_2t_3 + t_3t_1 + 2t_1t_2)(t_2t_3 + t_3t_1)}{(t_2 + t_3)(t_3 + t_1)(t_1 + t_2)} = \frac{(s_2 + t_1t_2)(s_2 - t_1t_2)}{s_1s_2 - s_3}$$

$$= \frac{s_2^2 - \dfrac{s_3^2}{t_3^2}}{s_1s_2 - s_3} = \frac{s_2^2}{s_1s_2 - s_3} - \frac{s_3^2}{(s_1s_2 - s_3)t_3^2}.$$

The three mid-points of the sides are therefore

$$x = \frac{s_2^2}{s_1s_2 - s_3} - \frac{s_3^2}{(s_1s_2 - s_3)t_i^2} \qquad i = 1, 2, 3.$$

Then

$$x = \frac{s_2^2}{s_1s_2 - s_3} - \frac{s_3^2}{s_1s_2 - s_3}t \tag{7.61}$$

† It is also sometimes called the *Lemoine* point or the *Grebe* point. See Johnson, Modern Geometry, p. 213.

is the map equation of the nine-point circle of the tangential triangle $P_1P_2P_3$. Note that its radius, $\left|s_3^2/(s_1s_2-s_3)\right|$, is just half the radius of the circumcircle (see 7.42) of this tangential triangle.

Writing the conjugate of 7.61,

$$y = \frac{s_1^2}{s_1s_2 - s_3} - \frac{1}{(s_1s_2 - s_3)t},$$

and eliminating t, we have for the self-conjugate equation of this nine-point circle

$$\left(x - \frac{s_2^2}{s_1s_2 - s_3}\right)\left(y - \frac{s_1^2}{s_1s_2 - s_3}\right) = \frac{s_3^2}{(s_1s_2 - s_3)^2}. \tag{7.62}$$

We now find the intersections of this circle with the circumcircle $xy=1$ of the triangle $t_1t_2t_3$. Substituting $1/x$ for y in 7.62, we have, after some reduction,

$$s_1^2x^2 - 2s_1s_2x + s_2^2 = 0.$$

If $t_1t_2t_3$ (and therefore also $P_1P_2P_3$) is equilateral, we have $s_1=s_2=0$, and the two circles coincide. Otherwise the quadratic equation has the double root $x=s_2/s_1$, which means that the nine-point circle of the tangential triangle $P_1P_2P_3$ is tangent to the circumcircle of the triangle $t_1t_2t_3$ at the point s_2/s_1.

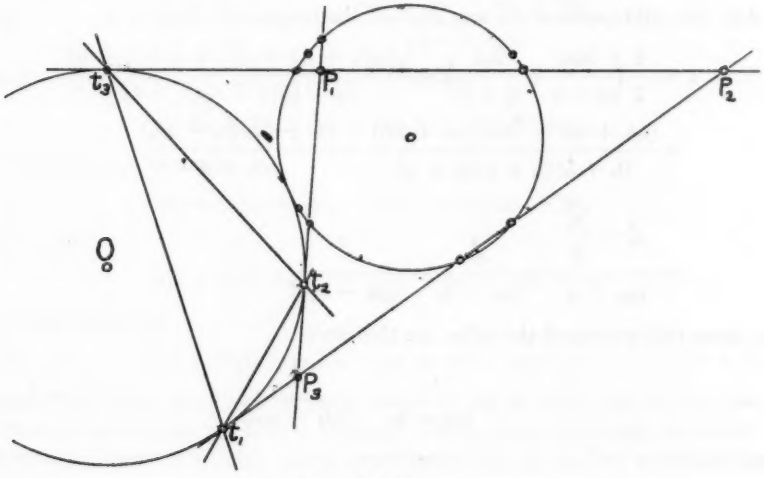

FIG. 20

We might start with any triangle $P_1P_2P_3$ (not equilateral) and take its incircle or any one of its excircles as the unit circle, with t_1, t_2, t_3 corresponding to the points of tangency of this circle with the lines P_2P_3, P_3P_1, and P_1P_2, and then

proceed with the foregoing proof. As a result we obtain what is known as *Feuerbach's theorem*:

THEOREM 7.63. *The nine-point circle of any triangle (not equilateral) is tangent to the incircle and to each of the excircles of the triangle.*

The corresponding facts for an equilateral triangle are quite obvious.

7.7. Consider now the question as to whether for a given triangle $P_1P_2P_3$ there is a circle with respect to which the triangle is self-polar (see section 4.3). If there is such a circle, its center must be the orthocenter of the triangle; and if the altitude from the vertex P_i meets the opposite side at the point D_i, each pair of inverse points, P_i, D_i, must lie on the same side of the orthocenter. This is possible only if the orthocenter is outside the triangle, and hence only an obtuse triangle can be self-polar with respect to any circle. We shall show that for any given obtuse triangle there is always a unique circle with respect to which the triangle is self-polar, and this circle may be called the *polar circle of the triangle.*

Fixing the coordinate system so that the vertices of the triangle are the points t_1, t_2, t_3 on the unit circle, the orthocenter will be the point s_1, and since the triangle is obtuse we have (see 7.32) $(s_1s_2-s_3)/s_3>0$. If there is a polar circle, its equation must be

$$(x - s_1)\left(y - \frac{s_2}{s_3}\right) = r^2, \tag{7.71}$$

where the value of r^2 is to be determined. The altitude from t_1 meets the opposite side at the point (see 7.22)

$$x = \frac{s_1}{2} - \frac{s_3}{2t_1^2},$$

and if this point and t_1 are to be inverse points with respect to the circle 7.71 we must have (see 4.21)

$$\left(\frac{s_1}{2} - \frac{s_3}{2t_1^2} - s_1\right)\left(\frac{1}{t_1} - \frac{s_2}{s_3}t_1\right) = r^2,$$

or

$$r^2 = \frac{s_3(-s_1t_1^2 + s_2t_1 - s_3) + s_1s_3t_1^3}{2s_3t_1^2}.$$

But we have $t_1^3-s_1t_1^2+s_2t_1-s_3=0$, and hence

$$-s_1t_1^2 + s_2t_1 - s_3 = -t_1^3$$

and

$$r^2 = \frac{-s_3 l_1^3 + s_1 s_2 l_1^3}{2s_3 l_1^3} = \frac{s_1 s_2 - s_3}{2s_3}.$$

Since $(s_1 s_2 - s_3)/s_3 > 0$, this value for r^2 is positive; and hence

$$(x - s_1)\left(y - \frac{s_2}{s_3}\right) = \frac{s_1 s_2 - s_3}{2s_3}$$

represents a circle with center at s_1, and having the vertex t_1 and the foot of the altitude from that vertex as inverse points. The symmetrical form of the result shows that each of the other vertices together with the foot of the corresponding altitude will also be a pair of inverse points, and we have

THEOREM 7.72. *If the triangle $t_1 t_2 t_3$ is obtuse, so that $(s_1 s_2 - s_3)/s_3 > 0$, it is a self-polar triangle with respect to the circle*

$$(7.73) \qquad (x - s_1)\left(y - \frac{s_2}{s_3}\right) = \frac{s_1 s_2 - s_3}{2s_3},$$

and this circle is the polar circle of the triangle.

This polar circle and three other circles that have been mentioned in connection with a triangle are circles of a pencil (see section 4.5). The four circles are the circumcircle, the nine-point circle 7.24, the polar circle 7.73 when it exists, and the circumcircle of the tangential triangle 7.44. We assume in this discussion that the triangle is not a right triangle and is not equilateral so that $s_1 s_2 - s_3 \neq 0$, $s_1 \neq 0$, $s_2 \neq 0$. The equations of these circles may be written respectively in the forms

$$(7.74) \qquad xy - 1 = 0,$$

$$(7.75) \qquad xy - \frac{s_2}{2s_3}x - \frac{s_1}{2}y + \frac{s_1 s_2 - s_3}{4s_3} = 0,$$

$$(7.76) \qquad xy - \frac{s_2}{s_3}x - s_1 y + \frac{s_1 s_2 + s_3}{2s_3} = 0,$$

$$(7.77) \qquad xy - \frac{2s_2}{s_1 s_2 - s_3}x - \frac{2s_1 s_3}{s_1 s_2 - s_3}y + \frac{4s_3}{s_1 s_2 - s_2} = 0.$$

It is readily verified that if we multiply 7.74 by -1 and 7.75 by 2 and add we get 7.76; and if we multiply 7.74 by $(s_1 s_2 - 5s_3)/(s_1 s_2 - s_3)$ and 7.75 by $4s_3/(s_1 s_2 - s_3)$ and add we get 7.77. Also, if we subtract any one of the equations from any other one, we obtain the linear equation

$$2s_2 x + 2s_1 s_2 y - s_1 s_2 - 3s_3 = 0.$$

This proves

THEOREM 7.78. *If the triangle $t_1t_2t_3$ is not a right triangle and is not equilateral, its circumcircle, nine-point circle, polar circle (when it exists), and the circumcircle of its tangential triangle all belong to a pencil whose radical axis is*

$$2s_3x + 2s_1s_2y - s_1s_2 - 3s_3 = 0.$$

The centers of these circles, as we have already seen, are on the Euler line (see 7.14, 7.21, and 7.43) $s_2x - s_1s_3y = 0$, perpendicular to the radical axis.

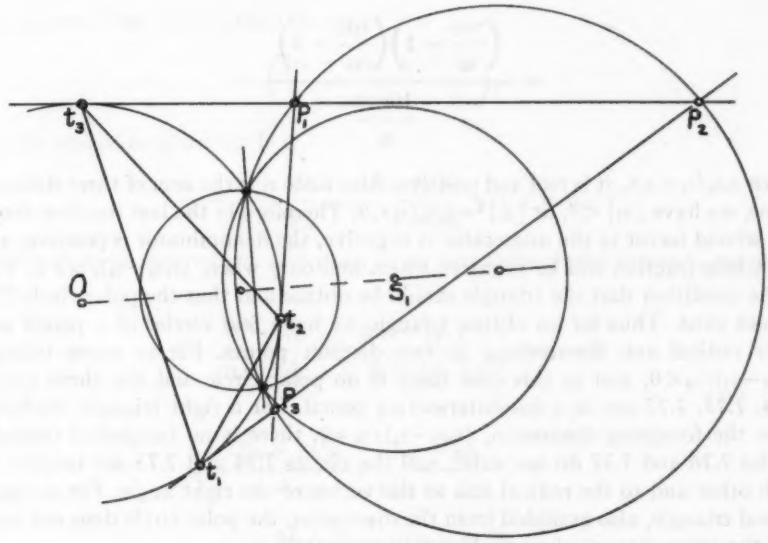

FIG. 21

The question as to whether these circles and their radical axis intersect or not can be answered by determining whether the radical axis cuts any one of the circles,* say the circle $xy-1=0$. The square of the distance from the center of this circle to the radical axis (see 3.55) is

$$d^2 = \frac{(s_1s_2 + 3s_3)^2}{16s_1s_2s_3},$$

* A natural procedure to determine whether or not a line intersects a circle would be to solve the equations simultaneously and then examine the pairs of values of x and y obtained as solutions to see if they are conjugates. But these solutions for x and y would be irrational in form, and it is rather difficult to determine whether or not *irrational* expressions are conjugates. In the case of a line cutting a circle it is simpler to use the distance from the center of the circle to the line.

and hence the axis cuts the circle in two points if

$$\frac{(s_1s_2+3s_3)^2}{16s_1s_2s_3}-1<0.$$

But

$$\frac{(s_1s_2+3s_3)^2}{16s_1s_2s_3}-1=\frac{s_1^2s_2^2-10s_1s_2s_3+9s_3^2}{16s_1s_2s_3}=\frac{(s_1s_2-s_3)(s_1s_2-9s_3)}{16s_1s_2s_3}$$

$$=\frac{\left(\frac{s_1s_2}{s_3}-1\right)\left(\frac{s_1s_2}{s_3}-9\right)}{\frac{16s_1s_2}{s_3}}.$$

Since $s_1s_2/s_3=s_1\bar{s}_1$, it is real and positive. Also since s_1 is the sum of three different turns, we have $|s_1|<3$, or $|s_1|^2=s_1s_2/s_3<9$. Therefore in the last fraction above the second factor in the numerator is negative, the denominator is positive, and the whole fraction will be negative when, and only when, $(s_1s_2-s_3)/s_3>0$. This is the condition that the triangle should be obtuse and that the polar circle 7.73 should exist. Thus for an obtuse triangle we have *four* circles of a pencil and their radical axis intersecting in two distinct points. For an acute triangle $(s_1s_2-s_3)/s_3<0$, and in this case there is no polar circle and the three circles 7.74, 7.75, 7.77 are in a non-intersecting pencil. For a right triangle, excluded from the foregoing discussion, $(s_1s_2-s_3)/s_3=0$, there is no tangential triangle, circles 7.76 and 7.77 do not exist, and the circles 7.74 and 7.75 are tangent to each other and to the radical axis at the vertex of the right angle. For an equilateral triangle, also excluded from the discussion, the polar circle does not exist and the other three circles are obviously concentric.

7.8. Three circles of another pencil are related to any triangle which is not isosceles or equilateral. They are called the *circles of Apollonius* or the *Apollonian circles*. One of these circles passes through the vertex t_1 and has its center at the intersection of the side t_2t_3 and the corresponding side P_2P_3 of the tangential triangle; and the other two circles are similarly related to the other two vertices and sides of the triangle. To find the equation of the circle through the vertex t_1 we note that the equations of the lines t_2t_3 and P_2P_3 are respectively (see 7.11 and 7.41)

$$\frac{x}{t_2t_3}+y=\frac{1}{t_2}+\frac{1}{t_3}\quad\text{and}\quad\frac{x}{t_1^2}+y=\frac{2}{t_1}.$$

Eliminating y and solving for x we have

$$x = \frac{t_1(2t_2t_3 - t_3t_1 - t_1t_2)}{t_2t_3 - t_1}$$
$$= \frac{t_1(3s_3 - s_2t_1)}{s_3 - t_1^3}.$$

Then, using the relation $t_1^3 - s_1t_1^2 + s_2t_1 - s_3 = 0$, we have

$$x = \frac{t_1(3s_3 - s_2t_1)}{s_2t_1 - s_1t_1^2} = \frac{3s_3 - s_2t_1}{s_2 - s_1t_1}.$$

The center of the circle is then the point

$$\left(\frac{3s_3 - s_2t_1}{s_2 - s_1t_1}, \frac{s_1 - 3t_1}{s_2 - s_1t_1}\right),$$

and the equation of the circle is

$$xy - \frac{s_1 - 3t_1}{s_2 - s_1t_1}x - \frac{3s_3 - s_2t_1}{s_2 - s_1t_1}y + g = 0,$$

where g is to be determined. Since the circle goes through the vertex t_1, we must have

$$1 - \frac{s_1 - 3t_1}{s_2 - s_1t_1}t_1 - \frac{3s_3 - s_2t_1}{s_2 - s_1t_1}\left(\frac{1}{t_1}\right) + g = 0,$$

or

$$g = \frac{(s_1 - 3t_1)t_1}{s_2 - s_1t_1} + \frac{3s_3 - s_2t_1}{t_1(s_2 - s_1t_1)} - 1$$
$$= \frac{-3t_1^3 + 2s_1t_1^2 - 2s_2t_1 + 3s_3}{t_1(s_2 - s_1t_1)}.$$

Using the identity $t_1^3 - s_1t_1^2 + s_2t_1 - s_3 = 0$, we have

$$g = \frac{-s_1t_1^2 + s_2t_1}{t_1(s_2 - s_1t_1)} = 1,$$

and the Apollonian circle corresponding to the vertex t_1 is therefore

$$xy - \frac{s_1 - 3t_1}{s_2 - s_1t_1}x - \frac{3s_3 - s_2t_1}{s_2 - s_1t_1}y + 1 = 0. \tag{7.81}$$

The equations of the three Apollonian circles may then be written

$$(s_2 - s_1t_i)xy - (s_1 - 3t_i)x - (3s_3 - s_2t_i)y + (s_2 - s_1t_i) = 0$$

or

$$(7.82) \qquad (s_2xy - s_1x - 3s_3y + s_3) - t_i(s_1xy - 3x - s_2y + s_1) = 0, \qquad i = 1, 2, 3.$$

That these circles belong to a pencil is seen by noting that the radical axis of *any two* of them is the line

$$(7.83) \qquad (s_1^2 - 3s_2)x - (s_2^2 - 3s_1s_3)y = 0,$$

and this radical axis is the Brocard line (see 7.55) through the circumcenter and the symmedian point.

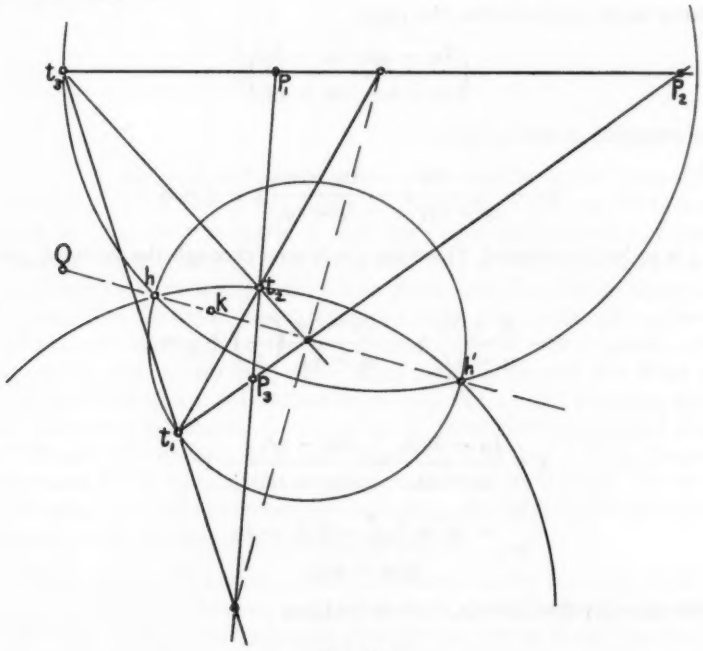

FIG. 22

That any two of the Apollonian circles must intersect in two points is intuitively obvious upon examination of the figure; and since the three circles belong to a pencil there must be two points common to all three circles and their radical axis. We shall find these two points in the next section, and show that their conjugate coordinates satisfy the equations*

$$(7.84) \qquad \begin{aligned} s_2xy - s_1x - 3s_3y + s_3 &= 0 \\ s_1xy - 3x - s_2y + s_1 &= 0, \end{aligned}$$

* These equations do not represent curves because they are not self-conjugate.

and hence the equations 7.82 of all three circles.

Since the Apollonian circles are in a pencil their centers must lie on a line. This line is called the *Lemoine line* of the triangle, and it is perpendicular to the radical axis of the pencil which is the Brocard line.

If the triangle is isosceles, say sides t_1t_2 and t_1t_3 are equal, we have $t_1^2 = t_2t_3$, or $s_2 - s_1t_1 = 0$. More generally, for any isosceles triangle we must have

$$(7.85) \qquad (s_2 - s_1t_1)(s_2 - s_1t_2)(s_2 - s_1t_3) = s_2^3 - s_1^3s_3 = 0.$$

In this case certain denominators in the preceding discussion will be zero, but the geometric facts in this special case are fairly obvious. If, for instance, sides t_1t_2 and t_1t_3 are equal, there will be no Apollonian circle through the vertex t_1; but the other two circles will exist and their radical axis, the axis of symmetry of the whole figure, will be both the Euler line and the Brocard line of the triangle. The reader may verify that when $s_2^3 - s_1^3s_3 = 0$, the equations 7.14 and 7.55 represent the same line.

7.9. In this section we shall assume that the vertices t_1, t_2, t_3, of our triangle are on the unit circle in *counter-clockwise order*, and that the triangle is not equilateral (Figure 23). Let us seek a point h whose reflections, h_1, h_2, h_3, in the sides t_2t_3, t_3t_1, t_1t_2, respectively are vertices of an equilateral triangle. Since h and h_1 are inverse points with respect to the side t_2t_3 with the equation (see 7.11)

$$\frac{x}{t_2t_3} + y = \frac{1}{t_2} + \frac{1}{t_3},$$

we must have (see 3.52)

$$\frac{h_1}{t_2t_3} + \bar{h} = \frac{1}{t_2} + \frac{1}{t_3},$$

or

$$h_1 = t_2 + t_3 - \bar{h}t_2t_3;$$

and similarly for h_2 and h_3. Then h_1, h_2, h_3 will be vertices of an equilateral triangle *in counter-clockwise* order (see 1.73) if

$$(t_2 + t_3 - \bar{h}t_2t_3) + \omega(t_3 + t_1 - \bar{h}t_3t_1) + \omega^2(t_1 + t_2 - \bar{h}t_1t_2) = 0,$$

or, using the relation $1 + \omega + \omega^2 = 0$ (see 1.51), if

$$\bar{h} = -\frac{t_1 + \omega t_2 + \omega^2 t_3}{t_2t_3 + \omega t_3t_1 + \omega^2 t_1t_2},$$

or if

$$(7.91) \qquad h = -\frac{t_2t_3 + \omega^2 t_3t_1 + \omega t_1t_2}{t_1 + \omega^2 t_2 + \omega t_3}.$$

If we set

$$p = t_1 + \omega t_2 + \omega^2 t_3, \qquad q = t_1 + \omega^2 t_2 + \omega t_3, \tag{7.92}$$

with

$$\bar{p} = \frac{t_2 t_3 + \omega^2 t_3 t_1 + \omega t_1 t_2}{s_3}, \qquad \bar{q} = \frac{t_2 t_3 + \omega t_3 t_1 + \omega^2 t_1 t_2}{s_3},$$

we can write

$$h = -\frac{\bar{p} s_3}{q}, \qquad \bar{h} = -\frac{p}{\bar{q} s_3}. \tag{7.93}$$

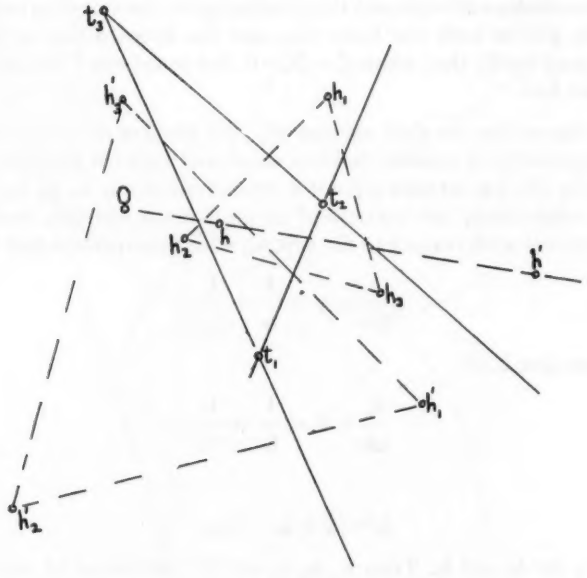

FIG. 23

Similarly, the reflections h'_1, h'_2, h'_3 of a point h' in the sides of the triangle will be vertices of an equilateral triangle *in clockwise order* if

$$h' = -\frac{\bar{q} s_3}{p}, \qquad \bar{h}' = -\frac{q}{\bar{p} s_3}. \tag{7.94}$$

These expressions for p and q and the coordinates of the points h and h' are not symmetric functions of t_1, t_2, t_3; but certain combinations of them are. The reader should verify the following relations.

$$
\begin{aligned}
p\bar{q} &= s_1^2 - 3s_2 \\
s_3^2 p\bar{q} &= s_2^2 - 3s_1 s_3 \\
s_3(p\bar{p} + q\bar{q}) &= 9s_3 - s_1 s_2 \\
h + h' &= -\frac{9s_3 - s_1 s_2}{s_1^2 - 3s_2} \\
hh' &= \frac{s_2^2 - 3s_1 s_3}{s_1^2 - 3s_2}.
\end{aligned} \tag{7.95}
$$

Hence, although h and h' separately are not rational symmetric functions of t_1, t_2, t_3, they are roots of a quadratic equation,

$$(s_1^2 - 3s_2)x^2 + (9s_3 - s_1 s_2)x + (s_2^2 - 3s_1 s_3) = 0,$$

whose coefficients are such rational symmetric functions.

The points h and h' are called the *isodynamic points* of the triangle $t_1 t_2 t_3$. The reader may verify readily that they are inverse points with respect to the circumcircle $xy - 1 = 0$. They lie on the Brocard line, and are in fact the points of intersection of the Apollonian circles (see section 7.8). These facts are established by showing that the coordinates $(h, \bar{h})$, and also $(h', \bar{h}')$, satisfy the equations (see 7.84)

$$
\begin{aligned}
s_2 xy - s_1 x - 3s_3 y + s_2 &= 0, \\
s_1 xy - 3x - s_3 y + s_1 &= 0.
\end{aligned}
$$

For $(h, \bar{h})$ we have

$$
\begin{aligned}
\frac{s_2 \bar{p}p}{\bar{q}q} + \frac{s_1 s_3 \bar{p}}{q} + \frac{3p}{\bar{q}} + s_2 &= \frac{s_2(\bar{p}p + \bar{q}q) + s_1 s_3 \bar{p}\bar{q} + 3pq}{q\bar{q}} \\
&= \frac{s_2(9s_3 - s_1 s_2) + s_1(s_2^2 - 3s_1 s_3) + 3s_3(s_1^2 - 3s_2)}{q\bar{q}s_3} = 0;
\end{aligned}
$$

and also

$$
\begin{aligned}
\frac{s_1 \bar{p}p}{\bar{q}q} + \frac{3s_3 \bar{p}}{q} + \frac{s_2 p}{s_3 \bar{q}} + s_1 &= \frac{s_1 s_3(\bar{p}p + \bar{q}q) + 3s_3^2 \bar{p}\bar{q} + s_2 pq}{\bar{q}qs_3} \\
&= \frac{s_1(9s_3 - s_1 s_2) + 3(s_2^2 - 3s_1 s_3) + s_2(s_1^2 - 3s_2)}{\bar{q}qs_3} = 0.
\end{aligned}
$$

The reader may verify similarly that the coordinates $(h', \bar{h}')$ satisfy the two equations.

We have then established the following theorems about the Apollonian

circles and isodynamic points of a triangle (see Figures 22 and 23):

THEOREM 7.96. *The three Apollonian circles of a triangle cut each other in the two isodynamic points.*

THEOREM 7.97. *The centers of the Apollonian circles (the points where the three sides of the triangle cut the three corresponding sides of the tangential triangle) lie on the Lemoine line.*

THEOREM 7.98. *The isodynamic points lie on the Brocard line of the triangle, and they are inverse points with respect to the circumcircle.*

THEOREM 7.99. *The reflections of each of the isodynamic points in the sides of the triangle are vertices of an equilateral triangle.*

Chapter VIII

THE PARABOLA

8.1. Let F be the focus of a parabola and H the reflection of the focus in the directrix. From any point P of the parabola drop the perpendicular PZ to the directrix. From the definition of a parabola $PZ=PF$, one sees that FPZ and

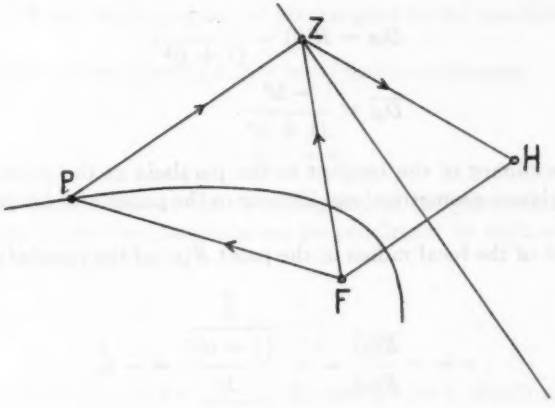

FIG. 24

FZH are similar isosceles triangles. If now in a system of conjugate coordinates, F and H correspond to the complex numbers f and h, and P and Z to the complex variables x and z, we see that vector $z-x$ is equal to the vector $x-f$ multiplied by a certain turn t, and vector $h-z$ is equal to vector $z-f$ multiplied by *the same turn* t. Hence we have

$$z - x = t(x - f)$$
$$h - z = t(z - f),$$

and eliminating z between these two equations we obtain

$$x = f + \frac{h - f}{(1 + t)^2} \tag{8.11}$$

as a general map equation of a parabola. For any parabola it is always possible to set up the coordinate system so that f and h are the points 0 and 1 respectively, and we then have the simplified map equation for the parabola

$$x = \frac{1}{(1 + t)^2}. \tag{8.12}$$

With this map equation the focus of the parabola is at the origin and the directrix is the line with the self-conjugate equation

$$x + y = 1. \tag{8.13}$$

The reader may verify that the self-conjugate equation of this parabola is

$$(x - y)^2 - 2(x + y) - 1 = 0.$$

From 8.12, writing $R(t)$ for $1/(1+t)^2$, we have

$$D_t x = R'(t) = \frac{-2}{(1 + t)^3},$$

$$\overline{D_t x} = \frac{-2t^3}{(1 + t)^3},$$

and hence the clinant of the tangent to the parabola at the point $R(t_1)$ is (see 5.42) t_1. This gives a geometrical significance to the parameter t in the map equation 8.12.

The clinant of the focal radius to the point $R(t_1)$ of the parabola is (see 3.42)

$$\tau = -\frac{\overline{R(t_1)}}{R(t_1)} = -\frac{\dfrac{t_1^2}{(1 + t_1)^3}}{\dfrac{1}{(1 + t_1)^2}} = -t_1^2,$$

and the clinant of the line through the point $R(t_1)$ parallel to the axis of reals is -1; therefore the clinants of the bisectors of the angles between these lines (see 3.32) are

$$\pm \sqrt{(-t_1^2)(-1)} = \pm t_1.$$

We have then the well known result.

THEOREM 8.14. *The tangent and normal at any point P of a parabola bisect the angles made by the focal radius FP and the line through P parallel to the axis of the parabola.*

The proofs of the two following theorems may be left as exercises for the reader.

THEOREM 8.15. *The points of tangency of two perpendicular tangents to a parabola are collinear with the focus.*

THEOREM 8.16. *A circle having a focal chord of a parabola as a diameter is tangent to the directrix.*

8.2. Consider now the map equation

$$x = \frac{1}{(1+t_1)(1+t)}, \tag{8.21}$$

in which t_1 is any *fixed* turn other than -1. This is a map equation of a straight line (see 3.11), and the reader can readily verify that the clinant of the line is t_1. But it is obvious that this line goes through the point $R(t_1)$ of the parabola when $t=t_1$; and since its clinant is the same as that of the tangent at $R(t_1)$, it follows that 8.21 is a map equation of the tangent to the parabola 8.12 at the point $R(t_1)$.

From the form of the equation 8.21 it is evident that the point

$$\frac{1}{(1+t_1)(1+t_2)} \tag{8.22}$$

is the point of intersection of the two tangents to the parabola at the points $R(t_1)$ and $R(t_2)$. If the two tangents are perpendicular to each other, $t_2=-t_1$, and the point of intersection is

$$x = \frac{1}{1-t_1^2}.$$

But when $x=1/(1-t_1^2)$, $y=\bar{x}=-t_1^2/(1-t_1^2)$, and $x+y=1$, which is the equation of the directrix (see 8.13). Hence

THEOREM 8.23. *The locus of intersections of pairs of perpendicular tangents to a parabola is the directrix.*

8.3. Consider the triangle formed by three tangents, l_i, with clinants t_i and map equations

$$x = \frac{1}{(1+t_i)(1+t)}, \qquad i = 1, 2, 3.$$

The vertex where l_2 and l_3 intersect is (see 8.22)

$$v_1 = \frac{1}{(1+t_2)(1+t_3)} = \frac{1+t_1}{(1+t_1)(1+t_2)(1+t_3)} = \frac{1+t_1}{1+s_1+s_2+s_3},$$

where s_1, s_2, s_3 are the symmetric functions of the three clinants. We shall write s for $1+s_1+s_2+s_3$, and note that since $t_i \neq -1$, $s \neq 0$; and also

$$\mathfrak{z} = 1 + \frac{s_2}{s_3} + \frac{s_1}{s_3} + \frac{1}{s_3} = \frac{s}{s_3}. \tag{8.31}$$

Then the three vertices of the triangle are

$$v_i = \frac{1+t_i}{s}, \qquad i = 1, 2, 3; \tag{8.32}$$

and these vertices evidently lie on the circle

$$x = \frac{1}{s} + \frac{1}{s}t, \tag{8.33}$$

which is the circumcircle of the triangle, with center at $1/s$ and radius $|1/s|$, and it passes through the origin when $t = -1$. Hence

THEOREM 8.34. *The circumcircle of a triangle formed by any three tangents to a parabola passes through the focus of the parabola.*

Now consider any four tangents to the parabola, l_i, with clinants t_i, $i = 1, 2, 3, 4$. Let Σ_1, Σ_2, Σ_3, Σ_4 be the symmetric functions of the four clinants, and s_1, s_2, s_3 the symmetric functions of three of them when any one, say t_k, is omitted. We have

$$s = 1 + s_1 + s_2 + s_3 = \frac{(1+t_1)(1+t_2)(1+t_3)(1+t_4)}{1+t_k}$$
$$= \frac{1+\Sigma_1+\Sigma_2+\Sigma_3+\Sigma_4}{1+t_k} = \frac{\Sigma}{1+t_k},$$

where Σ is written for $1+\Sigma_1+\Sigma_2+\Sigma_3+\Sigma_4$. Then the circumcenter of the triangle of three tangents, omitting the tangent l_k, is

$$\frac{1}{s} = \frac{1+t_k}{\Sigma}.$$

The four such circumcenters lie on the circle

$$x = \frac{1}{\Sigma} + \frac{1}{\Sigma}t, \tag{8.35}$$

and this circle also passes through the origin. We have then

THEOREM 8.36. *The circumcenters of the four triangles formed by any four tangents to a parabola lie on a circle which passes through the focus of the parabola.*

We now find the orthocenter of a triangle of three tangents. The vertex v_1 is (see 8.32) $(1+t_1)/s$, and the clinant of the altitude through this vertex is $-t_1$. Hence the equation of this altitude is

$$-t_1x + y = \frac{-t_1(1+t_1)}{s} + \frac{(1+t_1)s_3}{t_1s} = \frac{-t_1^3 + s_3 - t_1^2 + s_3t_1}{t_1s}.$$

Using the identity $t_1^3 - s_1 t_1^2 + s_2 t_1 - s_3 = 0$ (see 6.15) this becomes

$$-t_1 x + y = \frac{-s_1 t_1 + s_2 - t_1 + s_3}{s},$$

or

$$t_1(-sx + s_1 + 1) + (sy - s_2 - s_3) = 0.$$

Then the equations of the three altitudes are

$$t_i(-sx + s_1 + 1) + (sy - s_2 - s_3) = 0, \qquad i = 1, 2, 3,$$

and all three equations are obviously satisfied by the pair of *conjugate* values

$$x = \frac{1 + s_1}{s}, \qquad y = \frac{s_2 + s_3}{s}.$$

Hence the point

$$x = \frac{1 + s_1}{s} \tag{8.37}$$

is the orthocenter of the triangle. Adding the coordinates of this point we have

$$x + y = \frac{1 + s_1 + s_2 + s_3}{s} = 1,$$

and since $x+y=1$ is the equation of the directrix of the parabola (see 8.13) we have

THEOREM 8.38. *The orthocenter of any triangle formed by three tangents to a parabola lies on the directrix.*

8.4. By a *proper n-line* we shall mean a set of n lines no two of which are parallel and no three concurrent. Thus a proper 3-line is a triangle. It is well-known, from the point of view of projective geometry, that a proper 4-line determines a unique parabola to which each of the four lines is tangent. Hence for any proper 4-line the coordinate system may be set up in such a way that the four lines will be tangents to the parabola with the map equation 8.12, and therefore the preceding theorems about tangents to a parabola may be stated as theorems about any proper 4-line. Thus we have

THEOREM 8.41. *For any proper 4-line the circumcircles of the four triangles (omitting one line at a time) pass through a point called the node of the 4-line, this node being the focus of the parabola tangent to the four lines; and the four circumcenters of the triangles lie on a circle, the centric-circle of the 4-line, and this centric-circle also passes through the node.*

THEOREM 8.42. *For any proper 4-line the orthocenters of the four triangles (omitting one line at a time) lie on a line which is the directrix of the parabola tangent to the four lines.*

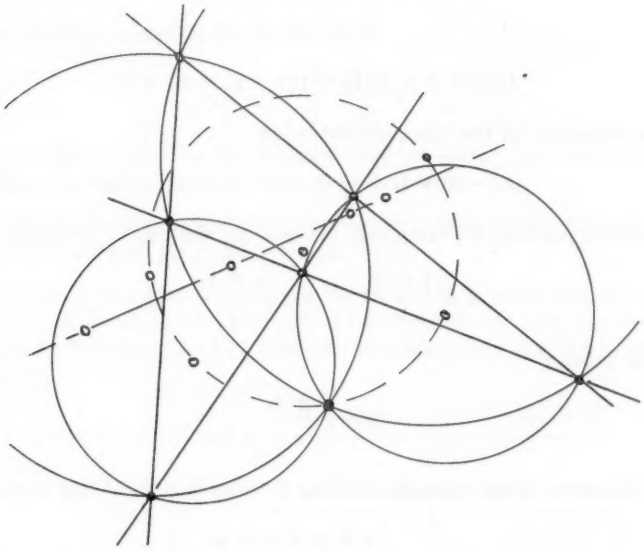

FIG. 25

We prove also the following

THEOREM 8.43. *If one line of a proper 4-line is parallel to the Euler line of the triangle formed by the other three lines, the same is true for every line of the 4-line.**

We regard the four lines as tangents to the parabola 8.12, map equations of the tangents being

$$x = \frac{1}{(1 + t_i)(1 + t)} \qquad i = 1, 2, 3, 4.$$

We have seen (see 8.33 and 8.37) that the circumcenter and orthocenter of a triangle of three tangents are respectively

$$\frac{1}{s} \quad \text{and} \quad \frac{1 + s_1}{s}.$$

* This problem, proposed by Howard Eves, appeared as problem 617 in the May, 1945, issue of the National Mathematics Magazine, and again as problem 54 in the January–February, 1950, issue of the Mathematics Magazine. A synthetic proof is given in a paper, The complete quadrilateral, by H. E. Fettis in the September–October, 1948, issue of the same magazine.

Then the clinant of the Euler line through these two points (see 3.42) is

$$-\frac{\frac{s_1}{s}}{\frac{s_1}{s}} = -\frac{\frac{s_2}{s}}{\frac{s_1}{s}} = -\frac{s_2}{s_1}.$$

Hence this Euler line is parallel to the remaining line if

$$-\frac{s_2}{s_1} = t_k,$$

or if

$$s_2 + s_1 t_k = 0.$$

But (see 6.21) $s_2 + s_1 t_k = \Sigma_2$, and the condition for parallelism becomes

$$\Sigma_2 = 0.$$

The symmetry of this result for the four lines proves the theorem.

Chapter IX

HYPOCYCLOIDS

9.1. If a circle rolls around a fixed base circle with no slipping at the point of contact, a point P on the circumference of the rolling circle describes a cycloidal curve, a *hypocycloid* if the rolling circle is inside the base circle and an *epicycloid* if the rolling circle is outside the base circle.

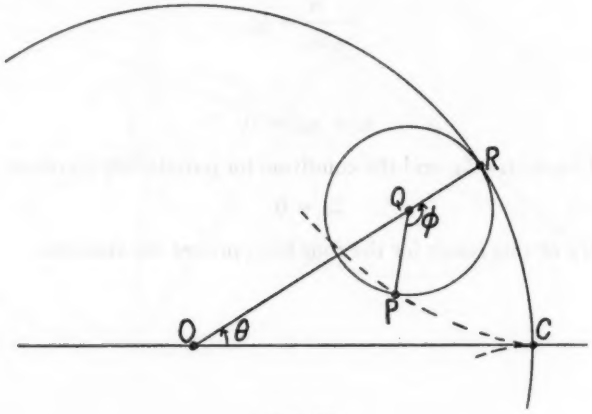

FIG. 26

Consider first the simple hypocycloid for which the radius of the base circle is n times that of the rolling circle, n an integer greater than 1. It is rather obvious that the point P describing the curve will come into contact with the base circle at n equally spaced points, resulting in a closed curve of n cusps.* We may think of the rolling motion as starting with the point P in contact with the base circle at a cusp C. We let r and r_1 be the radii of the base and rolling circles respectively, $r = nr_1$, and we set up the coordinate system with the origin O at the center of the base circle and the cusp C on the axis of reals at the point r. Let Q and R be respectively the moving center of the rolling circle and the moving point of contact of the two circles; and let θ be the variable angle from the vector OC to the vector OR, and ϕ the variable angle from the vector QP to the vector QR. Since the arc CR of the base circle must be of the same length as the arc PR of the rolling circle, we have $r\theta = r\phi/n$, or $\phi = n\theta$. Then

$$\text{vector } OP = \text{vector } OQ + \text{vector } QP,$$

* The reader is probably familiar with the hypocycloid of four cusps whose equation in Cartesian coordinates is

$$x^{2/3} + y^{2/3} = a^{2/3}.$$

$$\text{vector } OQ = \left(r - \frac{r}{n}\right)e^{i\theta} = \frac{r}{n}(n-1)e^{i\theta},$$

$$\text{vector } QP = \frac{r}{n}e^{-i(\phi-\theta)} = \frac{r}{n}e^{-i\theta(n-1)}.$$

If now we put $e^{i\theta} = t$, and let x be the complex number corresponding to the point P (and therefore also to the vector OP), we have

$$x = \frac{r}{n}\left\{(n-1)t + \frac{1}{t^{n-1}}\right\}, \qquad n > 1, \tag{9.11}$$

as a map equation of the hypocycloid of n cusps. Letting $\epsilon = e^{2\pi i/n}$, so that ϵ^s, $s = 1, 2, \cdots, n$, are the nth roots of unity (see section 1.5), it is geometrically obvious that the cusps occur at the points $r\epsilon^s$, $s = 1, 2, \cdots, n$. These are the points where the derivative $D_t x$ vanishes; for we have from 9.11,

$$D_t x = \frac{r(n-1)(t^n - 1)}{nt^n},$$

and this derivative vanishes when $t = \epsilon^s$. The corresponding points of the curve are

$$x = \frac{r}{n}\left\{(n-1)\epsilon^s + \frac{1}{\epsilon^{s(n-1)}}\right\} = \frac{r}{n}\left\{(n-1)\epsilon^s + \left(\frac{1}{\epsilon^{n-1}}\right)^s\right\}$$

$$= \frac{r}{n}\{(n-1)\epsilon^s + \epsilon^s\} = r\epsilon^s,$$

and hence the cusps occur at the points where $D_t x = 0$ (see section 5.4).

We now consider some special cases of interest.

9.2. The map equation of the hypocycloid of two cusps is

$$x = \frac{r}{2}\left(t + \frac{1}{t}\right). \tag{9.21}$$

Since t and $1/t$ are conjugates, their sum is real, and the point x moves back and forth on the axis of reals from $x = r$ when $t = 1$ to $x = -r$ when $t = -1$. Thus the hypocycloid of two cusps is a line segment of length $2r$, the "cusps" in this case being the ends of the segment. The derivative $D_t x$ vanishes when $t = \pm 1$, the values of t that give the ends of the segment.

If t is eliminated from the equation 9.21 and the conjugate equation

$$y = \frac{r}{2}\left(\frac{1}{t} + t\right),$$

we get the self-conjugate equation $x - y = 0$ of the axis of reals. It should be noted that this self-conjugate equation is satisfied not only by all the points of the

segment given by the map equation 9.21 but also by points of the axis of reals outside this segment (see 5.13).

The more general map equation (see 5.21)

$$x = a + b\left(t + \frac{1}{t}\right), \tag{9.22}$$

where $b = \rho e^{i\theta} \neq 0$, represents a line segment of length 4ρ, making an angle θ with the axis of reals, and having its center at the point a.

9.3. The hypocycloid of three cusps is known as the *deltoid* and its map equation is

$$x = \frac{r}{3}\left(2t + \frac{1}{t^2}\right).$$

In order to study the properties of any one such deltoid, we may choose our unit of length in such a way that $r = 3$, and we then have the simpler map equation

$$x = 2t + \frac{1}{t^2}, \tag{9.31}$$

and in the study of the deltoid we shall write $R(t)$ for $2t + 1/t^2$. For this curve we have

$$D_t x = 2 - \frac{2}{t^3},$$

$$\overline{D_t x} = 2 - 2t^3,$$

and hence the clinant of the tangent to the curve at the point $R(t_1)$ (see 5.42) is

$$\frac{2 - 2t_1^3}{t_1^2\left(2 - \frac{2}{t_1^3}\right)} = -t_1.$$

It follows that no two tangents to the deltoid can be parallel.

The cusp radius (radius of the base circle) is 3, and the cusps are at the points 3, 3ω, and $3\omega^2$, given by the values 1, ω, and ω^2 for t. The reader may verify the fact (which is rather obvious mechanically) that the deltoid is tangent to the unit circle at the points -1, $-\omega$, and $-\omega^2$.

Consider now the map equation

$$x = t_1 + t + \frac{1}{t_1 t}, \tag{9.32}$$

in which t_1 is a fixed turn. It is obvious that when t takes on the value t_1 the deltoid 9.31 and the curve 9.32 have the point $R(t_1)$ in common. Moreover, at this common point the tangents to the two curves have the same direction; because for the curve 9.32,

$$D_t x = 1 - \frac{1}{t_1 t^2},$$

$$\overline{D_t x} = 1 - t_1 t^2,$$

and the clinant of the tangent at the point where $t = t_1$ is

$$\frac{1 - t_1^3}{t_1^2\left(1 - \frac{1}{t_1^3}\right)} = -t_1,$$

the same as the clinant of the tangent to the deltoid at the common point. Hence for any fixed value of the turn t_1, the map equation 9.32 represents a curve tangent to the deltoid at the point $R(t_1)$. For different values of t_1 we have a family of curves one of which is tangent to the deltoid at each point of the deltoid. We call this family of curves, given by the map equation 9.32, *penosculants** of the deltoid. The similar penosculants of the parabola (see section 8.2) were the straight lines tangent to the parabola.

Letting T_1 represent either one of the square roots of t_1, so that $T_1^2 = t_1$, the equation 9.32 may be written

$$x = t_1 + \frac{1}{T_1}\left(T_1 t + \frac{1}{T_1 t}\right).$$

Comparing this with 9.22, and noting that $T_1 t$ is a turn (see section 5.2), we see that the penosculant is a hypocycloid of two cusps, a line segment, 4 units in length and with its center at the point t_1 on the unit circle. If $T_1 = e^{i\phi}$ (and hence $t_1 = e^{2i\phi}$), the segment makes an angle $-\phi$ with the axis of reals. The ends of the segment are the points $x = t_1 \pm 2/T_1 = T_1^2 \pm 2/T_1$, but these are the points $R(\pm 1/T_1)$ *on the deltoid*. We have then proved

THEOREM 9.33. *If at any point of a deltoid a tangent line segment is drawn with its ends on the deltoid, the mid-point of the segment will lie on the inscribed circle of the deltoid and its length will be four times the radius of this circle.*

The results of this paragraph may be stated in another way. If the center of a line segment 4 units long moves around a circle of unit radius in the counter-clockwise direction while the segment itself rotates about its center half as fast in the clockwise direction, the segment will describe a deltoid *in three distinct*

* The name penosculant was introduced by Morley because of the analogy with Study's theory of osculants; *cf.* Morley, Metric geometry of the plane n-line, Trans. Amer. Math. Soc., vol. 1, 1900, p. 97.

ways—each end of the segment will describe the deltoid and the segment will envelop the deltoid.

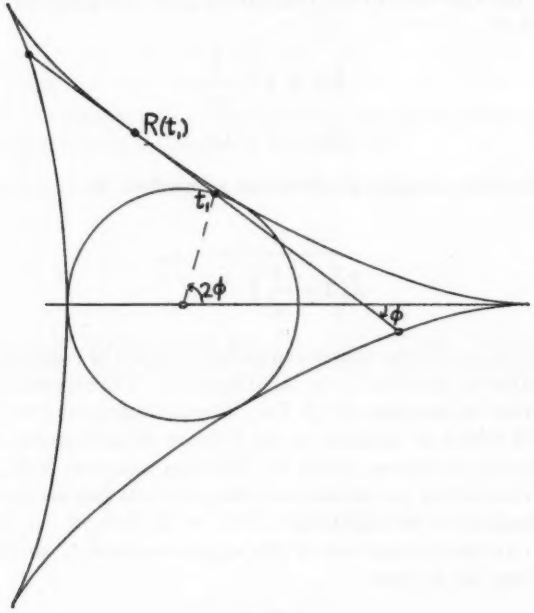

FIG. 27

9.4. The method of the preceding section may be extended to prove a remarkable theorem* about a simple hypocycloid of any number of cusps.

The map equation of a hypocycloid of n cusps, H_n, is

$$x = (n-1)t + \frac{1}{t^{n-1}} = R(t),$$

and it has a family of penosculant curves with map equations

$$x = t_1 + (n-2)t + \frac{1}{t_1 t^{n-2}}. \tag{9.41}$$

For any fixed t_1 the penosculant curve is tangent to the hypocycloid H_n at the point $R(t_1)$. If now T_1 is any one of the $(n-1)$th roots of t_1, so that $T_1^{n-1} = t_1$, the map equation of the penosculant curve may be written

* This theorem was given in a paper, On adjustable cycloidal and trochoidal curves, by Morley, Amer. J. of Math., vol. 16, 1894, p. 188.

$$x = t_1 + \frac{1}{T_1}\left\{(n-2)T_1 t + \frac{1}{(T_1 t)^{n-2}}\right\}. \tag{9.42}$$

Since $T_1 t$ is a turn, we see that this is the map equation (see 5.21) of a hypocycloid of $n-1$ cusps, H_{n-1}, with its center at the point t_1 of the unit circle. Its $n-1$ cusps occur for those values of t for which $D_t x = 0$. From equation 9.41,

$$D_t x = n - 2 - \frac{n-2}{t_1 t^{n-1}} = (n-2)\left(1 - \frac{1}{t_1 t^{n-1}}\right),$$

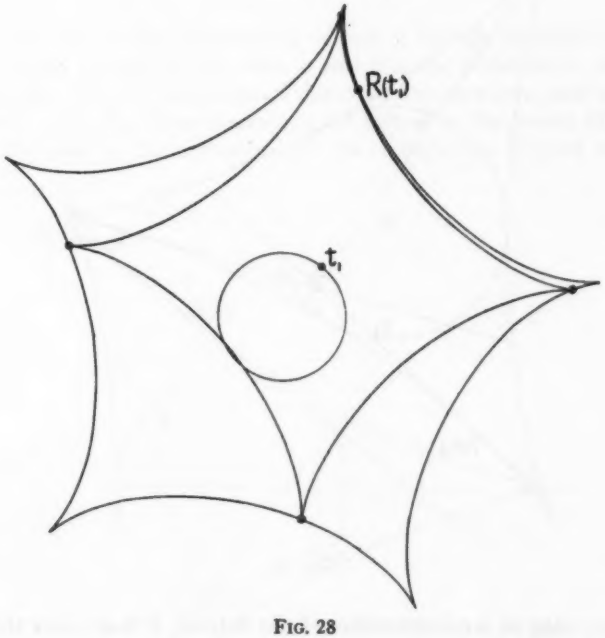

FIG. 28

and $D_t x$ vanishes when t is any one of the $(n-1)$th roots of $1/t_1$, that is, when $t = 1/T_1$. Putting this value of t in 9.41, we see that there is a cusp of H_{n-1} at the point

$$x = t_1 + (n-2)\frac{1}{T_1} + \frac{T_1^{n-2}}{t_1} = T_1^{n-1} + (n-2)\frac{1}{T_1} + \frac{1}{T_1}$$

$$= (n-1)\frac{1}{T_1} + \frac{1}{\left(\frac{1}{T_1}\right)^{n-1}}.$$

But this is the point $R(1/T_1)$ on the hypocycloid H_n; and hence each cusp of the penosculant hypocycloid H_{n-1} lies on the hypocycloid H_n. As the turn t_1 varies, the penosculant hypocycloid rotates with its $n-1$ cusps always on the hypocycloid H_n, its center on the unit circle, and always tangent to the hypocycloid H_n at the point $R(t_1)$. Figure 28 shows the case $n=5$. Theorem 9.33 is merely the special case for $n=3$ of this more general theorem.

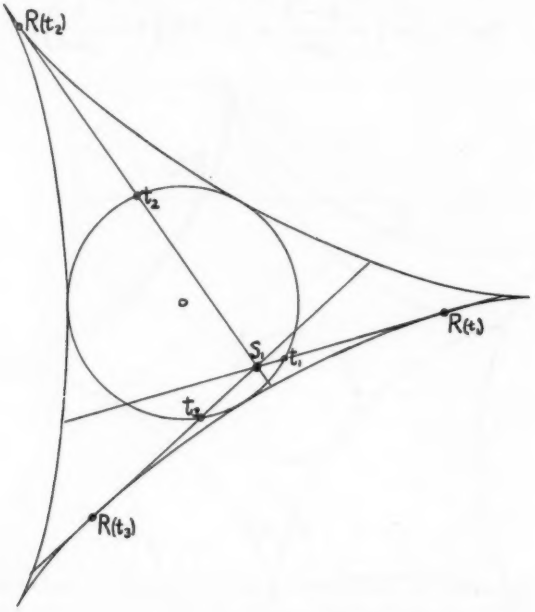

FIG. 29

9.5. Returning to a consideration of the deltoid, it is obvious that the two tangent line segments

$$x = t_1 + t + \frac{1}{t_1 t} \quad \text{and} \quad x = t_2 + t + \frac{1}{t_2 t}$$

intersect at the point

$$x = t_1 + t_2 + \frac{1}{t_1 t_2}.$$

The clinants of these two tangents (see section 9.3) are $-t_1$ and $-t_2$, and hence they are perpendicular if $t_2 = -t_1$. In this case their point of intersection is $x = -1/t_1^2$, a point on the unit circle. Thus the locus of intersections of pairs of

perpendicular tangents to the deltoid is the unit circle, the inscribed circle of the deltoid.

Three tangents which do not meet in a point form a triangle with vertices

$$x = s_1 - t_i + \frac{t_i}{s_3}, \qquad i = 1, 2, 3.$$

Then

$$x = s_1 + \left(\frac{1}{s_3} - 1\right)t \tag{9.51}$$

is the map equation of the circumcircle of such a triangle of tangents. The circumcenter of the triangle is the point s_1 which is the orthocenter (see 7.13) of the triangle $t_1t_2t_3$. If $s_3 = 1$, the radius of this circle becomes zero; and hence under the condition $s_3 = 1$, the three tangents to the deltoid at the points $R(t_i)$, $i = 1, 2, 3$, meet at the point s_1, the orthocenter of the triangle $t_1t_2t_3$ (Figure 29).

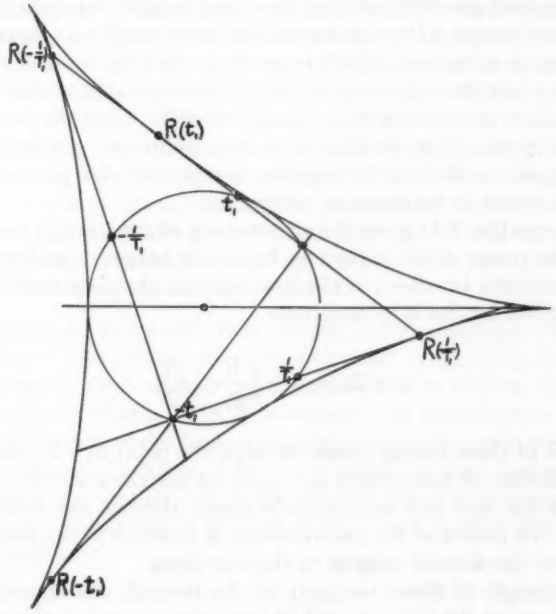

FIG. 30

The reader may show that the self-conjugate equation of the tangent to the deltoid $x = 2t + 1/t^2$ at the point where $t = t_1$ is

$$t_1^3 - t_1^2 x + t_1 y - 1 = 0,$$

and if three tangents meet at the point (x_1, y_1), the clinants of the tangents will be the three roots of the equation

$$t^3 + x_1t^2 + y_1t + 1 = 0.$$

We have seen that the ends of the line segment

$$x = t_1 + t + \frac{1}{t_1t}$$

tangent to the deltoid at the point $R(t_1)$, are on the deltoid at the points $R(\pm 1/T_1)$, where $T_1^2 = t_1$. The clinants of the tangents at these two ends are $\mp 1/T_1$, and hence the two tangents are perpendicular to each other. Moreover, these tangents intersect at the point $-T_1^2 = -t_1$ on the unit circle, and the third tangent through this point is the tangent at $R(-t_1)$, perpendicular to the tangent at (Rt_1) and having $-t_1$ as the mid-point of the segment (Figure 30).

9.6. It has been pointed out that there is a unique parabola tangent to the four lines of any proper 4-line (no two of the lines parallel, no three concurrent). Similarly there is a unique deltoid tangent to the four lines of any proper 4-line, and hence any theorem about a proper 4-line may be proved by assuming that the four lines are all tangent to the deltoid 9.31. Thus the theorems proved in section 8.4 by regarding the four lines as tangents to a parabola may also be proved by regarding the lines as tangents to a deltoid. We prove Theorem 8.41 and leave the others as exercises for the reader.

The map equation 9.51 gives the circumcircle of the triangle formed by three tangents at the points $R(t_i)$. If then we have four tangents, and write Σ_1, Σ_2, Σ_3, Σ_4 for the symmetric functions of the four turns t_i, the circumcircles of the four triangles are given by the map equations

$$x = \Sigma_1 - t_i + \left(\frac{t_i}{\Sigma_4} - 1\right)t, \qquad i = 1, 2, 3, 4.$$

For $t = \Sigma_4$ each of these circles passes through the point $\Sigma_1 - \Sigma_4$, the node of the 4-line. Also the four circumcenters, $\Sigma_1 - t_i$, lie on the circle $x = \Sigma_1 - t$, the centric-circle of the 4-line, and this centric-circle passes through the node when $t = \Sigma_4$.

Note that the radius of the centric-circle is 1, which is the radius of the inscribed circle of the deltoid tangent to the four lines.

Given a triangle of three tangents to the deltoid, the clinant of the side tangent at the point $R(t_1)$ is $-t_1$ and the opposite vertex is the point $t_2 + t_3 + 1/(t_2t_3)$, and therefore the equation of the corresponding altitude is

$$t_1x + y = t_1t_2 + t_1t_3 + \frac{t_1}{t_2t_3} + \frac{1}{t_2} + \frac{1}{t_3} + t_2t_3$$

or

$$t_1 x + y = s_2 + \frac{s_1 t_1}{s_3}.$$

The three such altitudes have the equations

$$t_i x + y = s_2 + \frac{s_1 t_i}{s_3}, \qquad i = 1, 2, 3,$$

and they obviously meet at the point $(s_1/s_3, s_2)$; and hence the orthocenter of a triangle of three tangents is the point s_1/s_3. We have seen that the circumcenter of this triangle is the point s_1. Since s_3 is a turn, $|s_3| = 1$ and $|s_1/s_3| = |s_1|$, and it follows that the circumcenter and orthocenter of any triangle of three tangents to a deltoid are equidistant from the center of the deltoid. This gives a method of locating the center of the deltoid tangent to the lines of a given proper 4-line. For each of the four triangles we draw the perpendicular bisector of the segment from the circumcenter to the orthocenter, and these four lines must meet at the center of the tangent deltoid. The size of the deltoid is fixed by the fact that the centric-circle of the 4-line is the same size as the inscribed circle of the deltoid, and the base circle of the deltoid is three times as large. Thus, given a proper 4-line, the inscribed circle and the base circle of the deltoid touching the four lines can be readily constructed. But there remains the rather difficult problem of locating a cusp of the deltoid on the base circle. For the reader who may wish to work at this problem it should be pointed out that the locating of a cusp involves the trisection of a general angle. The trisection of an angle presents no *practical* difficulty in making a drawing, but it cannot be accomplished by a strict compass and straight-edge construction.

9.7. We have considered only those simple hypocycloids for which the ratio r/r_1 is an integer. The reader may be interested in proving that when $r/r_1 = n/(n-1)$ we get the same *simple* hypocycloid as when $r/r_1 = n$. Thus the same deltoid will be described by a point on a rolling circle whose radius is either one-third or two-thirds that of the base circle. However, if the ratio r/r_1 is a rational fraction, $r/r_1 = p/q$, p, q integers, $q \neq p-1$, a nodal (or star) hypocycloid is obtained. Replacing n in equation 9.11 by p/q will give a map equation for such a curve. The star hypocycloid of 5 cusps, with $r/r_1 = 5/2$, is an interesting curve to study and to draw.

Chapter X

EPICYCLOIDS AND TROCHOIDS

10.1. As noted in section 9.1, if a circle rolls around a fixed base circle, the rolling circle *externally* tangent to the base circle, a point P on the circumference of the rolling circle describes a curve called an *epicycloid*. For the *simple* epicycloid the radius of the base circle is n times that of the rolling circle, n any positive integer. As in the case of the hypocycloid, it is obvious that the point P comes in contact with the base circle at n equally spaced points, these points in this case being cusps pointing in toward the center of the base circle.

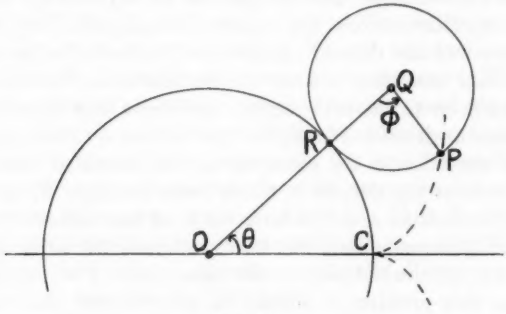

FIG. 31

Again we let r and r_1 be the radii of the base and rolling circles respectively, $r = nr_1$, and we set up the coordinate system with the origin O at the center of the base circle and a cusp C on the axis of reals at the point r. With Q as the center of the rolling circle, R the point of contact with the base circle, θ the angle from the vector OC to the vector OR, and ϕ the angle from the vector QR to the vector QP, we have

$$n\theta = \phi,$$

$$\begin{aligned} \text{vector } OP &= \text{vector } OQ + \text{vector } QP \\ &= \frac{r}{n}(n+1)e^{i\theta} + \frac{r}{n}e^{i(\theta+\phi-\pi)} \\ &= \frac{r}{n}(n+1)e^{i\theta} - \frac{r}{n}e^{i\theta(n+1)}. \end{aligned}$$

Then writing r for $e^{i\theta}$ we have the map equation of the curve

$$x = \frac{r}{n}\{(n+1)t - t^{n+1}\}. \tag{10.11}$$

It may be noted that this is the map equation we would get if we would replace n in equation 9.11 by $-n$. Thus if we allow n to take on negative as well as positive integral values, either equation 9.11 or equation 10.11 will include all simple hypocycloids and epicycloids of $|n|$ cusps.

10.2. We could not have a hypocycloid of one cusp, but the epicycloid of one cusp is the familiar *cardioid* whose equation in polar coordinates is $\rho = a(1-\cos\theta)$. Putting $n=1$ in 10.11 we have

$$x = r(2t - t^2);$$

or, if we take the radius of the base circle as the unit of length, we have for the cardioid the simple map equation

$$x = 2t - t^2. \tag{10.21}$$

In the study of the properties of this curve in this section we may write $R(t)$ for $2t-t^2$.

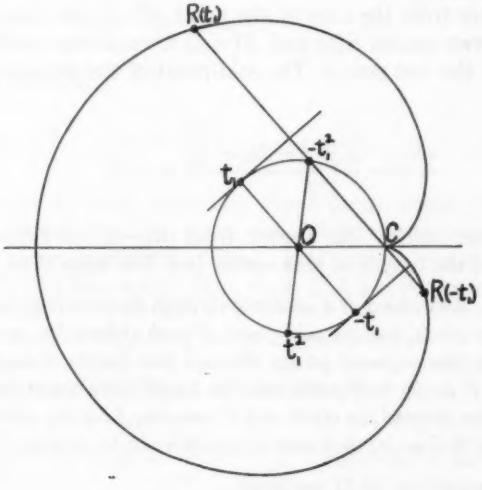

FIG. 32

Consider the reflections of the point 1 in the tangents to the unit circle. The equation of the tangent to the unit circle at the point t_1 (see 7.41) is

$$\frac{1}{t_1^2}x + y - \frac{2}{t_1} = 0$$

or

$$x + t_1^2 y - 2t_1 = 0.$$

Then the reflection of the point 1 in this tangent (see 3.54) is

$$x_2 = 2t_1 - t_1^2,$$

and this is the point $R(t_1)$ on the cardioid $x = R(t)$. Hence we have

THEOREM 10.22. *If C is a fixed point on a circle, the locus of the reflections of C in the tangents to the circle is a cardioid with its cusp at C and the given circle as its base circle.*

We see also that the perpendicular bisector of the line segment from the cusp to the point $R(t_1)$ of the cardioid $x = R(t)$ is tangent to the base circle at the point t_1, and this gives a simple graphical method of finding the turn t_1 which corresponds to any point $R(t_1)$ of the cardioid. Moreover the vector from the cusp to the point $R(t_1)$, $t_1 \neq 1$, is $2t_1 - t_1^2 - 1 = t_1(2 - t_1 - 1/t_1)$. But $t_1 + 1/t_1$ is real and less than 2, and hence $2 - t_1 - 1/t_1$ is positive. Hence the vector from the cusp to the point $R(t_1)$ has the same direction (arrowhead direction) as the vector t_1. Also the vector from the cusp to the point $R(-t_1)$ has the opposite direction, and thus the two points $R(t_1)$ and $R(-t_1)$ are collinear with the cusp, the cusp lying between the two points. The mid-point of the segment from $R(t_1)$ to $R(-t_1)$ is

$$\frac{(2t_1 - t_1^2) + (-2t_1 - t_1^2)}{2} = -t_1^2,$$

a point on the base circle. The vector from $R(-t_1)$ to $R(t_1)$ is $(2t_1 - t_1^2) - (-2t_1 - t_1^2) = 4t_1$, and the length of this vector is 4. We have then

THEOREM 10.23. *Any chord of a cardioid through the cusp is of length four times the radius of the base circle, and the mid-point of such a chord lies on the base circle; and, conversely, if a line segment passes through two points C and P of a circle, the segment having P as its mid-point and its length four times the radius of the circle, then as P moves around the circle and C remains fixed the ends of the segment describe a cardioid with C as its cusp and the circle as its base circle.*

From the map equation 10.21 we have

$$D_t x = 2(1 - t),$$

and hence the clinant of the tangent to the cardioid at the point $R(t_1)$ (see 5.42) is

$$\tau = \frac{2\left(1 - \frac{1}{t_1}\right)}{2t_1^2(1 - t_1)} = -\frac{1}{t_1^3}.$$

The equation of this tangent is then

$$-\frac{1}{t_1^2}x + y = -\frac{2t_1 - t_1^3}{t_1^2} + \frac{2t_1 - 1}{t_1^2}$$

or

$$t_1^3 y - 3t_1^2 + 3t_1 - x = 0. \tag{10.24}$$

We call t_1 the *parameter* of this tangent at the point $R(t_1)$. Because of the relation 10.24 it follows that if any tangent of the cardioid passes through a given point (x_1, y_1) of the plane, the parameter of this tangent must satisfy this equation in t:

$$t^3 - \frac{3}{y_1}t^2 + \frac{3}{y_1}t - \frac{x_1}{y_1} = 0.$$

If this equation is satisfied by three turns, t_1, t_2, t_3, we must have

$$s_1 = \frac{3}{y_1}, \qquad s_2 = \frac{3}{y_1}, \qquad s_3 = \frac{x_1}{y_1},$$

from which we must have

$$s_1 = s_2 \neq 0, \qquad x_1 = \frac{3s_3}{s_1}, \qquad y_1 = \frac{3}{s_2} = \frac{3}{s_1}, \qquad y_1 = \bar{x}_1.$$

Suppose now that we start with any three turns which satisfy the relation $s_1 = s_2 \neq 0$, and let $x_1 = 3s_3/s_1$, $y_1 = 3/s_1 = \bar{x}_1$. Then the tangents to the cardioid $x = R(t)$ at the three points $R(t_1)$, $R(t_2)$, $R(t_3)$ will be concurrent at the point $(3s_3/s_1, 3/s_1)$. For the tangent at $R(t_i)$, $i = 1, 2, 3$, has the equation

$$t_i^3 y - 3t_i^2 + 3t_i - x = 0$$

and the point $(3s_3/s_1, 3/s_1)$ lies on this tangent since

$$\frac{3t_i^3}{s_1} - 3t_i^2 + 3t_i - \frac{3s_3}{s_1} = \frac{3}{s_1}(t_i^3 - s_1 t_i^2 + s_2 t_i - s_3) = 0.$$

Thus we have the following theorems:

THEOREM 10.25. *If t_1, t_2, t_3 are three turns such that $s_1 = s_2 \neq 0$, then the tangents to the cardioid $x = 2t - t^3$ at the points $R(t_1)$, $R(t_2)$, $R(t_3)$ are concurrent at the point $(3s_3/s_1,\ 3/s_1)$.*

THEOREM 10.26. *If three tangents to the cardioid $x = 2t - t^3$ are concurrent at the point (x_1, y_1), the parameters of the tangents are the three roots of the equation*

$$y_1 t^3 - 3t^2 + 3t - x_1 = 0.$$

The reader may prove also that through a point (not the cusp) of the cardioid $x=2t-t^2$ two tangents will pass, the tangent at the point and a second tangent whose parameter is $(2t_1-t_1^2)/(2t_1-1)$.

If we have three turns of the form t_1, ωt_1, $\omega^2 t_1$, we have at once $s_1=s_2=0$, the case excluded in theorem 10.25. Since the clinant of the tangent at the point $R(t_1)$ is $-1/t_1^3$, it follows that the tangents at the three points $R(t_1)$, $R(\omega t_1)$ and $R(\omega^2 t_1)$ have the same clinant. Also the vectors t_1, ωt_1, and $\omega^2 t_1$ make angles of 120° with each other, and the same must be true of the vectors from the cusp to the three points of contact. Hence we have

THEOREM 10.27. *For each direction in the plane there are three parallel tangents to the cardioid, and the vectors from the cusp to the three points of contact make angles of* 120° *with each other.*

As special cases of this theorem, the three tangents parallel to the axis of reals have the points of contact $R(1)=1$, $R(\omega)=2\omega-\omega^2=1+3\omega$, and $R(\omega^2)=2\omega^2-\omega^4=1+3\omega^2$; while the three tangents perpendicular to the axis of reals have the points of contact $R(-1)=-3$, $R(-\omega)=-2\omega-\omega^2=1-\omega$, and $R(-\omega^2)=-2\omega^2-\omega^4=1-\omega^2$. In the latter case the tangents at $1-\omega$ and $1-\omega^2$ coincide as the double tangent of the cardioid with the equation $x+y=3$.

Associated with the cardioid $x=2t-t^2$ is a family of penosculant curves with the map equation

$$x = t_1 + t - t_1 t = t_1 + (1 - t_1)t. \tag{10.28}$$

As in preceding cases (see sections 8.2, 9.3, 9.4) it is easily verified that for any fixed t_1 the penosculant curve is tangent to the cardioid at the point $R(t_1)$. Also the penosculant curve is obviously a circle with its center at the point t_1 and passing through the cusp (the point 1) when $t=1$. Thus we have

THEOREM 10.29. *A circle passing through the cusp of the cardioid $x=2t-t^2$ and having its center at the point t_1 on the base circle will be tangent to the cardioid at the point $R(t_1)$; and, conversely, if a family of circles have their centers on a fixed base circle and all pass through a fixed point C of this base circle, they will envelop a cardioid with its cusp at C.*

10.3. Two penosculant circles, C_1 and C_2, of the cardioid $x=2t-t^2$ with their centers at t_1 and t_2 evidently intersect at the cusp and at the point $t_1+t_2-t_1t_2$. For three circles with centers at t_1, t_2, t_3 the three intersections (other than the cusp) will be

$$x_i = s_1 - t_i - \frac{s_3}{t_i}, \qquad i = 1, 2, 3,$$

and these three intersections lie on the curve with the map equation

$$x = s_1 - t - \frac{s_3}{t}.$$

If T is one of the square roots of s_2, $T^2 = s_2$, this equation may be written

$$x = s_1 - T\left(\frac{t}{T} + \frac{T}{t}\right). \tag{10.31}$$

Comparing this with 9.22 and noting that t/T is a turn, one sees that we have the map equation of a line segment 4 units in length and center at the point s_1. If now we consider four penosculant circles, we will have four such line segments l_i, the line l_1 passing through the intersections of the circles C_2, C_3, C_4, *etc.* The intersection of circles C_3 and C_4 is the intersection of lines l_1 and l_2, and C_1 is the circumcircle of the triangle formed by lines l_2, l_3, and l_4. We have then again the figure of the proper 4-line (see sections 8.4 and figure, and 9.6), with circles C_i as the circumcircles of the four triangles, the cusp C as the node, and the base circle of the cardioid as the centric-circle of the 4-line. Conversely, if we start with any given proper 4-line, the associated cardioid is uniquely determined since its cusp and base circle are the node and centric-circle of the 4-line. In the cases of the parabola and deltoid associated with a 4-line, the four lines were tangent to the curves, but in the case of the cardioid it is the four circumcircles which are tangent to the curve.

The reader may be interested in drawing an accurate figure of a proper 4-line, its four circumcircles, its node and centric-circle, and the associated parabola, deltoid, and cardioid. As it is rather difficult to locate the deltoid when the 4-line is given, it is best to start with a deltoid, using four of its tangents for the 4-line. Then the parabola is fixed by its focus and directrix and the cardioid by its base circle and cusp.

10.4. If a circle rolls around inside or outside a fixed base circle (Figure 33), a point P attached to the rolling circle, but not necessarily on its circumference, describes a curve called a *trochoid*. Hypocycloids and epicycloids are included in the more general class of trochoids. Simple trochoids are those for which the radius r of the base circle is $|n|$ times the radius r_1 of the rolling circle, n an integer positive or negative but not zero.

Let the point P be attached to the rolling circle at a distance kr_1 from its center, $0 < k \leqq 1$*. Then by the same procedure that was used to obtain map equations 9.11 and 10.11 we find for the general map equation for all simple trochoids

$$x = \frac{r}{n}\left[(n-1)t + \frac{k}{t^{n-1}}\right]. \tag{10.41}$$

This equation gives hypo-trochoids for n positive and epi-trochoids for n negative (see section 10.1), and it includes the simple hypocycloids and epicycloids

* Any curve that can be constructed with $k > 1$ can also be constructed with a value of $k < 1$ and a different r and r_1.

when $k=1$. The *simple* trochoids ($r/r_1=n$ or $n/(n-1)$) may be convex curves for values of k near 0, but for values of k near 1 they have rounded points which are nearly cusps.

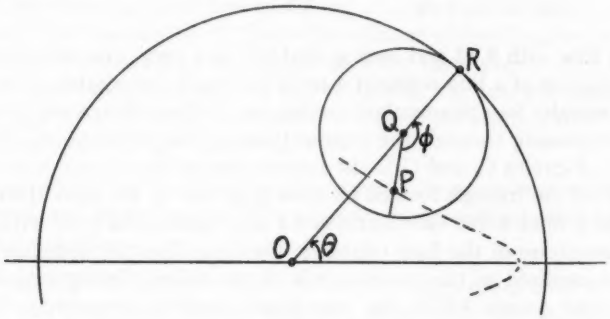

FIG. 33

10.5. We examine the rather special hypo-trochoid for $n=2$, $0<k<1$, with the map equation

$$x = \frac{r}{2}\left[t + \frac{k}{t}\right] = R(t). \tag{10.51}$$

We shall show that this curve is an ellipse with foci at the points $\pm r\sqrt{k}$ on the axis of reals. The vector from the point $r\sqrt{k}$ to the point $R(t_1)$ of the curve is

$$\frac{r}{2}\left[t_1 + \frac{k}{t_1}\right] - r\sqrt{k}$$

or

$$\frac{r}{2t_1}(t_1 - \sqrt{k})^2.$$

The length of this vector is

$$\sqrt{\frac{r}{2t_1}(t_1 - \sqrt{k})^2 \cdot \frac{rt_1}{2}\left(\frac{1}{t_1} - \sqrt{k}\right)^2} = \frac{r}{2t_1}(t_1 - \sqrt{k})(1 - t_1\sqrt{k}).$$

Similarly the distance from the point $-r\sqrt{k}$ to the point $R(t_1)$ is

$$\frac{r}{2t_1}(t_1 + \sqrt{k})(1 + t_1\sqrt{k}),$$

and the sum of these distances is

$$\frac{r}{2t_1}(2t_1 + 2kt_1) = r(1 + k).$$

Since this sum is a constant for all t_1, the curve must be an ellipse with $\pm r\sqrt{k}$ as its foci. Putting $t = \pm 1$ in the map equation 10.51 we see that the ellipse cuts the axis of reals at the points $\pm\frac{1}{2}r(1+k)$, and for $t = \pm i$ it cuts the axis of pure imaginaries at the points $\pm\frac{1}{2}r(1-k)i$, so that the semi-major and minor axes are

$$a = \frac{r}{2}(1 + k), \qquad b = \frac{r}{2}(1 - k).$$

From these relations we have

$$r = a + b, \qquad k = \frac{a - b}{a + b},$$

giving the values of r and k which correspond to an ellipse of any specified dimensions.

Writing the map equation of the ellipse in the form

$$x = \frac{t^2 + k}{2t},$$

we have a family of penosculant curves

$$x = \frac{t_1 t + k}{t_1 + t}.$$

The reader may verify that these penosculants are straight lines (see section 5.3) tangent to the ellipse.

A line segment through the center of an ellipse with its ends on the curve is called a *diameter* of the ellipse. From the form of the map equation 10.51 it is obvious that the points $R(\pm t_1)$ are the ends of a diameter.

From 10.51 we have

$$D_t x = \frac{r}{2}\left(1 - \frac{k}{t^2}\right),$$

and hence the clinant of the tangent to the ellipse at each of the points $R(\pm t_1)$ is

$$\tau = \frac{\frac{r}{2}(1 - kt_1^2)}{\frac{r}{2}\left(1 - \frac{k}{t_1^2}\right)t_1^2} = \frac{kt_1^2 - 1}{k - t_1^2}. \tag{10.52}$$

The clinant of the focal radius from the focus $r\sqrt{k}$ to the point $R(t_1)$ of the ellipse is

$$-\frac{\frac{r}{2}\left(\frac{1}{t_1}+kt_1\right)-r\sqrt{k}}{\frac{r}{2}\left(t_1+\frac{k}{t_1}\right)-r\sqrt{k}}=-\frac{(1-t_1\sqrt{k})^2}{(t_1-\sqrt{k})^2}.$$

Similarly the clinant of the focal radius from the focus $-r\sqrt{k}$ to the same point is

$$-\frac{(1+t_1\sqrt{k})^2}{(t_1+\sqrt{k})^2}.$$

The clinants of the bisectors of the angles between these focal radii (see 3.32) are

$$\pm\sqrt{\frac{(1-t_1\sqrt{k})^2(1+t_1\sqrt{k})^2}{(t_1-\sqrt{k})^2(t_1+\sqrt{k})^2}}=\pm\frac{1-t_1^2k}{t_1^2-k},$$

and comparing this with 10.52 we see that these are the clinants of the tangent and normal at the point. This proves the well known

THEOREM 10.53. *The tangent and normal at any point of an ellipse bisect the angles made by the focal radii through the point.*

Chapter XI

THE RECTANGULAR HYPERBOLA

11.1. It is easily verified (see section 5.3) that the map equations

$$x = \frac{i+t}{1+t} \quad \text{and} \quad x = \frac{1+t}{i+t}$$

represent two perpendicular lines, l_1 and l_2, through the origin, with self-conjugate equations $ix+y=0$ and $-ix+y=0$ respectively. Consider now the map equation

$$x = \frac{i+t}{1+t} + \frac{1+t}{i+t}. \tag{11.11}$$

We shall show that this equation represents a rectangular hyperbola with the lines l_1 and l_2 as asymptotes. If we write z for $(i+t)/(1+t)$, we see that z is a variable point on the line l_1, or the vector from the origin to this point; and similarly $1/z$ is the vector from the origin to the point $1/z$ on the line l_2; and the sum of these two vectors is the vector x from the origin to the point x of the curve. But the product of the lengths of the vectors z and $1/z$ is

$$\sqrt{z\bar{z}} \cdot \sqrt{\frac{1}{z} \cdot \frac{1}{\bar{z}}} = 1,$$

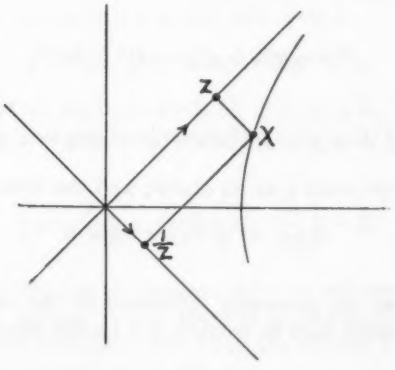

FIG. 34

and since this product is a constant the curve must be a rectangular hyperbola with the lines l_1 and l_2 as asymptotes. The reader may verify that the vertices of the hyperbola are the points $\pm\sqrt{2}$ on the axis of reals, and the foci are the points ± 2.

Since

$$\bar{z} = \frac{\dfrac{1}{i} + \dfrac{1}{t}}{1 + \dfrac{1}{t}} = \frac{i + t}{i(1 + t)} = \frac{z}{i},$$

we may write equation 11.11 and its conjugate as

$$x = z + \frac{1}{z} \quad \text{and} \quad y = \frac{z}{i} + \frac{i}{z},$$

and eliminating z we have

(11.12) $$x^2 + y^2 = 4 \quad \text{or} \quad (ix + y)(-ix + y) = 4$$

as the self-conjugate equation of this particular rectangular hyperbola. For a general map-equation of *any* rectangular hyperbola (see 5.21) we have

$$x = a + c\left(z + \frac{1}{z}\right), \qquad a, c \text{ arbitrary complex constants,}$$

and from this we deduce the general self-conjugate equation

$$\left(\frac{x - a}{c}\right)^2 + \left(\frac{y - \bar{a}}{\bar{c}}\right)^2 = 4,$$

or

(11.13) $$\bar{c}^2(x - a)^2 + c^2(y - \bar{a})^2 = 4c^2\bar{c}^2,$$

or

(11.14) $$\{\bar{c}(x - a) + ic(y - \bar{a})\}\{\bar{c}(x - a) - ic(y - \bar{a})\} = 4c^2\bar{c}^2.$$

This hyperbola has the point a as its center, and the lines

$$\bar{c}(x - a) \pm ic(y - \bar{a}) = 0$$

as its asymptotes.

In this chapter we are primarily interested in the *rectangular* hyperbola; but the general hyperbola may be treated in a similar manner, the general map equation being

$$x = a + c\left(\frac{t_0 + t}{1 + t} + \frac{1 + t}{t_0 + t}\right).$$

11.2. Four points, of which no three are collinear, ordinarily determine a unique rectangular hyperbola which passes through them, but there are two exceptional cases. If the set of points is an orthocentric set (see section 7.2)

there are infinitely many rectangular hyperbolas through them, a case which will be treated in the next section. If the set of points is not orthocentric, but if the line through one pair of them is perpendicular to the line through the other pair, there is, strictly speaking, no rectangular hyperbola through the points; but the pair of perpendicular lines through them may be regarded as a degenerate case of a rectangular hyperbola.

We find now the equation of the rectangular hyperbola through four points of the unit circle. Equation 11.13 may be written in the form

$$\bar{c}^2x^2 - 2a\bar{c}^2x + (a^2\bar{c}^2 + \bar{a}^2c^2 - 4c^2\bar{c}^2) - 2\bar{a}c^2y + c^2y^2 = 0, \tag{11.21}$$

and if we solve this simultaneously with the equation of the unit circle $xy=1$, eliminating y, we have

$$\bar{c}^2x^4 - 2a\bar{c}^2x^3 + (a^2\bar{c}^2 + \bar{a}^2c^2 - 4c^2\bar{c}^2)x^2 - 2\bar{a}c^2x + c^2 = 0.$$

If the hyperbola passes through four points t_i, $i=1, 2, 3, 4$, of the unit circle, these four turns t_i are the roots of the above quartic in x; and if we write Σ_1, Σ_2, Σ_3, Σ_4 for the elementary symmetric functions of the t_i, we must have

$$\begin{cases} \Sigma_1 = 2a, & \Sigma_3 = \dfrac{2\bar{a}c^2}{\bar{c}^2}, \\ \Sigma_2 = \dfrac{a^2\bar{c}^2 + \bar{a}^2c^2 - 4c^2\bar{c}^2}{\bar{c}^2}, & \Sigma_4 = \dfrac{c^2}{\bar{c}^2}. \end{cases} \tag{11.22}$$

Putting these results in equation 11.21 we see that the equation of the rectangular hyperbola through four points t_i of the unit circle is

$$x^2 - \Sigma_1x + \Sigma_2 - \Sigma_3y + \Sigma_4y^2 = 0. \tag{11.23}$$

Also from relations 11.22 we have $a=\Sigma_1/2$, $c/\bar{c} = \pm\sqrt{\Sigma_4}$, and hence it follows that the center of the rectangular hyperbola 11.23 is $\Sigma_1/2$, and its asymptotes are

$$\left(x - \frac{\Sigma_1}{2}\right) \pm i\sqrt{\Sigma_4}\left(y - \frac{\Sigma_3}{2\Sigma_4}\right) = 0. \tag{11.24}$$

Four points of a circle can obviously not be an orthocentric set; but it may happen that the line joining two of them is perpendicular to the line joining the other two. The line through t_1 and t_2 has the clinant $1/t_1t_2$ (see section 7.1), and hence this special case will occur when

$$\left(\frac{1}{t_1t_2} + \frac{1}{t_3t_4}\right)\left(\frac{1}{t_1t_3} + \frac{1}{t_2t_4}\right)\left(\frac{1}{t_1t_4} + \frac{1}{t_2t_3}\right) = 0,$$

or when

$$(t_3t_4 + t_1t_2)(t_2t_4 + t_1t_3)(t_2t_3 + t_1t_4) \equiv \Sigma_1^2\Sigma_4 + \Sigma_3^2 - 4\Sigma_2\Sigma_4 = 0.$$

But equation 11.23 may be written

$$\left\{\left(x-\frac{\Sigma_1}{2}\right)+i\sqrt{\Sigma_4}\left(y-\frac{\Sigma_3}{2\Sigma_4}\right)\right\}\left\{\left(x-\frac{\Sigma_1}{2}\right)-i\sqrt{\Sigma_4}\left(y-\frac{\Sigma_3}{2\Sigma_4}\right)\right\}$$

$$=\frac{\Sigma_1^2\Sigma_4+\Sigma_3^2-4\Sigma_2\Sigma_4}{4\Sigma_4},$$

and hence in this exceptional case equation 11.23 represents a pair of perpendicular lines, the lines given by equations 11.24.

11.3. We may now study the family of rectangular hyperbolas passing through the vertices of any triangle. We take the circumcircle of the triangle as the unit circle, the vertices corresponding to three turns, t_1, t_2, t_3. A hyperbola cutting a circle in three points must cut it also in a fourth point, though this fourth point may coincide with one of the three points. Therefore we get *all* rectangular hyperbolas through the vertices of the triangle if we consider the rectangular hyperbolas determined by the vertices and a variable fourth point on the unit circle. For convenience in what follows we take the fourth point as t^2 where t is a variable turn. The hyperbolas will then be given by the equation 11.23 if the Σ's are symmetric functions of t_1, t_2, t_3, t^2. If we write s_1, s_2, s_3 for the symmetric functions of t_1, t_2, t_3 and use the relations 6.21, equation 11.23 becomes

$$(11.31)\qquad x^2-(s_1+t^2)x+(s_2+s_1t^2)-(s_3+s_2t^2)y+s_3t^2y^2=0,$$

and this is then the equation of the family of all rectangular hyperbolas through the vertices of the triangle.

The coordinates of the orthocenter of the triangle (see 7.13) are $(s_1, s_2/s_3)$, and substituting these coordinates for x and y in equation 11.31 we find that the equation is satisfied identically. Thus all the rectangular hyperbolas through the vertices of a triangle pass also through the orthocenter; or, to state it more symmetrically, all rectangular hyperbolas through three points of an orthocentric set pass also through the fourth point. This is the exceptional case, mentioned in the preceding section, of a set of four points which do not determine a *unique* rectangular hyperbola.

For any fixed value of t the center of the hyperbola is

$$a=\frac{s_1+t^2}{2};$$

and hence the locus of the centers of all the hyperbolas is the circle whose map equation is

$$x=\frac{s_1}{2}+\frac{t^2}{2},$$

and this is the nine-point circle (see 7.21) of the triangle.

Referring to 11.24 we see that the asymptotes of the hyperbola 11.31 for a fixed t are

$$\left(x - \frac{s_1 + t^2}{2}\right) \pm i\sqrt{s_3 t^2}\left(y - \frac{s_3 + s_2 t^2}{2s_3 t^2}\right) = 0.$$

If we let T be any one of the sixth roots of s_3, so that

$$T^6 = s_3, \qquad \pm\sqrt{s_3 t^2} = \pm T^3 t,$$

the equation of one asymptote* may be written

$$\left(x - \frac{s_1 + t^2}{2}\right) + iT^3 t\left(y - \frac{s_3 + s_2 t^2}{2s_3 t^2}\right) = 0$$

or

$$\frac{x}{iT^3 t} + y - \frac{s_1}{2iT^3 t} - \frac{t}{2iT^3} - \frac{1}{2t^2} - \frac{s_2}{2s_3} = 0. \tag{11.32}$$

We wish to obtain the equation of the envelope of these asymptotes as t varies, and the usual procedure would be to differentiate 11.32 partially with respect to the parameter t and then eliminate t between the equation thus obtained and equation 11.32. But since the partial differentiation with respect to t gives us an equation independent of y, we may solve at once for x in terms of t to obtain a map equation of the envelope. Thus, differentiating 11.32 partially with respect to t, we have

$$-\frac{x}{iT^3 t^2} + \frac{s_1}{2iT^3 t^2} - \frac{1}{2iT^3} + \frac{1}{t^3} = 0,$$

from which we obtain

$$x = \frac{s_1}{2} + \frac{iT^3}{t} - \frac{t^2}{2}$$

or

$$x = \frac{s_1}{2} + \frac{T^2}{2}\left\{2\left(\frac{iT}{t}\right) + \left(\frac{t}{iT}\right)^2\right\}. \tag{11.33}$$

If we compare this result with 9.31 and note that iT/t runs through all turn values as t runs through all turn values, we see that this envelope of the asymptotes is a deltoid with center at $s_1/2$, the nine-point center of the triangle, and having the nine-point circle as its inscribed circle.

* For any value of t, say $t = t_0$, equation 11.31 represents a certain hyperbola and equation 11.32 one of its asymptotes; and when $t = -t_0$, 11.31 gives the *same* hyperbola and 11.32 its *other* asymptote. As t runs through all turn values we get each hyperbola twice but each asymptote only once.

The line given by equation 11.32 for any particular value of t, say $t=t_0$ touches the deltoid at the point

$$x = \frac{s_1}{2} + \frac{iT^3}{t_0} - \frac{t_0^2}{2}.$$

For three special values of t^2 equation 11.31 represents two perpendicular lines rather than a proper hyperbola, namely, when $t^2 = -s_3/t_j^2$, $j=1, 2, 3$. Each of these pairs of lines consists of a side of the triangle and the corresponding altitude. These sides and altitudes are included in the family of lines given by equation 11.32, the sides when $t=-iT^3/t_j$ and the altitudes when $t=iT^3/t_j$. Thus the sides of the triangle are tangent to the deltoid at the points

$$\frac{s_1}{2} - t_j + \frac{s_3}{2t_j^2},$$

and the altitudes at the points

$$\frac{s_1}{2} + t_j + \frac{s_3}{2t_j^2}.$$

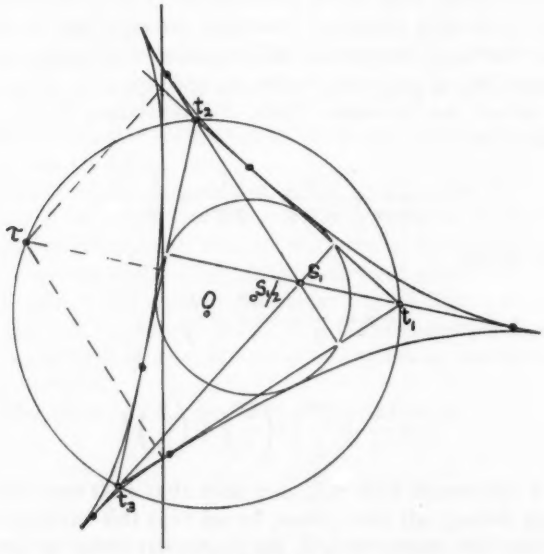

FIG. 35

Since four tangents determine a deltoid uniquely, this is the only deltoid tangent to the sides and altitudes of the triangle. Thus we have proved

THEOREM 11.34. *All the rectangular hyperbolas through the vertices of a triangle pass also through the orthocenter; the locus of their centers is the nine-point circle of the triangle, and the envelope of their asymptotes is the deltoid which is tangent to the sides and altitudes of the triangle and has the nine-point circle as its inscribed circle.*

11.4. The lines of theorem 11.34 which are the asymptotes of the rectangular hyperbolas and the tangents to the deltoid are also related to the triangle in another way. If from any point τ of the circumcircle of the triangle $t_1t_2t_3$ perpendiculars are dropped to the three sides, the feet of these perpendiculars will lie on a line. To see this we note that the equation of the side t_2t_3 of the triangle (see 7.11) is

$$\frac{x}{t_2t_3} + y = \frac{1}{t_2} + \frac{1}{t_3}$$

and the line perpendicular to this side from the point τ is

$$-\frac{x}{t_2t_3} + y = \frac{-\tau}{t_2t_3} + \frac{1}{\tau}.$$

Solving these equations for the point of intersection we have

$$x_1 = \frac{1}{2}\left(t_2 + t_3 + \tau - \frac{t_2t_3}{\tau}\right)$$

or

$$x_1 = \frac{1}{2}\left(s_1 + t_1 + \tau - \frac{s_3}{t_1\tau}\right).$$

Then the three feet of the perpendiculars from the point τ to the three sides will lie on the curve whose map equation is

$$x = \frac{1}{2}\left(s_1 + \tau - t - \frac{s_3}{\tau t}\right). \tag{11.41}$$

Following the method that was used in the case of equation 10.31, one could show that this is the map equation of a line segment (hypocycloid of two cusps). But it will suit our present purpose better to obtain the self-conjugate equation of the line by eliminating the parameter t between equation 11.41 and the conjugate equation

$$y = \frac{1}{2}\left(\frac{s_2}{s_3} + \frac{1}{\tau} - \frac{1}{t} - \frac{\tau t}{s_3}\right).$$

The resulting linear equation is

$$-\frac{\tau x}{s_3}+y+\frac{s_1\tau}{2s_3}+\frac{\tau^2}{2s_3}-\frac{s_2}{2s_3}-\frac{1}{2\tau}=0. \tag{11.42}$$

For any point τ on the circumcircle of a triangle there is such a line, called the Simson* line for the point τ. As τ varies we have a whole family of such Simson lines given by equation 11.42, and we shall now show that these Simson lines are exactly the asymptotes of the hyperbolas of theorem 11.34. If we simply substitute in equation 11.32 giving the asymptotes and in equation 11.42 giving the Simson lines, remembering that $T^6=s_3$, we see that the asymptote for $t=iT^3/\tau$ is the Simson line for τ, and the Simson line for $\tau=iT^3/t$ is the asymptote for t. It follows that the Simson lines for any triangle are the tangents to the deltoid of theorem 11.34; and for this reason the deltoid is sometimes called the *Simson deltoid* (or Simson quartic) of the triangle.

* There is considerable doubt as to whether Simson was the discoverer of these lines. See the historical note in Johnson's Modern Geometry, p. 137, and the paper by Mackay to which Johnson refers.

REFERENCES

The following are some of the papers which have made use of the conjugate coordinate system.

F. MORLEY

1. On the epicycloid, Amer. Jour. of Math., vol. 13, 1891, pp. 179–204.
2. Adjustable cycloidal and trochoidal curves, Amer. Jour. of Math., vol. 16, 1894, pp. 188–204.
3. Metric geometry of the plane n-line, Trans. Amer. Math. Soc., vol. 1, 1900, pp. 97–115.
4. Orthocentric properties of the plane n-line, Trans. Amer. Math. Soc., vol. 4, 1903, pp. 1–12.
5. On reflexive geometry, Trans. Amer. Math. Soc., vol. 8, 1907, pp. 14–24.
6. Extensions of Clifford's chain theorem, Amer. Jour. of Math., vol. 51, 1929, pp. 465–472.

F. H. LOUD

7. Sundry metric theorems concerning n lines in a plane, Trans. Amer. Math. Soc., vol. 1, 1900, pp. 323–338.

R. P. STEPHENS

8. On the pentadeltoid, Trans. Amer. Math. Soc., vol. 7, 1906, pp. 207–227.
9. On a system of parastroids, Annals of Math., vol. 8, ser. 2, 1907, pp. 159–171.

H. A. CONVERSE

10. On a system of hypocycloids of class three, Annals of Math., vol. 5, ser. 2, 1904, pp. 105–139.

C. S. ATCHISON

11. Curves with a directrix, Johns Hopkins University dissertation, 1908.

J. E. HODGSON

12. Orthocentric properties of the plane directed n-line, Trans. Amer. Math. Soc., vol. 13, 1912, pp. 199–231.

W. B. CARVER

13. The failure of the Clifford chain, Amer. Jour. of Math., vol. 42, 1920, pp. 137–157.

P. S. WAGNER

14. An extension to Clifford's chain, Amer. Jour. of Math., vol. 51, 1929, pp. 473–481.

J. R. MUSSELMAN

15. On equilateral triangles, this MONTHLY, vol. 39, 1932, pp. 290–292.
16. On certain types of polygons, this MONTHLY, vol. 40, 1933, pp. 157–160.
17. The triangle bordered with squares, this MONTHLY, vol. 43, 1936, pp. 539–548.
18. On four lines and their associated parabola, this MONTHLY, vol. 44, 1937, pp. 513–521.
19. On the line of images, this MONTHLY, vol. 45, 1938, pp. 421–430.
20. On the geometry of the triangle, this MONTHLY, vol. 46, 1939, pp. 347–348.
21. Some loci connected with a triangle, this MONTHLY, vol. 47, 1940, pp. 354–361.

F. V. MORLEY

22. Note on the incenters of a quadrilateral, this MONTHLY, vol. 27, 1920, pp. 252–255.
23. The three-bar curve, this MONTHLY, vol. 31, 1924, pp. 71–77.

J. H. WEAVER

24. Curves determined by a one-parameter family of triangles, this MONTHLY, vol. 40, 1933, pp. 85–91.
25. On isogonal points, this MONTHLY, vol. 42, 1935, pp. 496–499.
26. A generalization of the circles of Apollonius, this MONTHLY, vol. 45, 1938, pp. 17–21.
27. Some properties of lines associated with a parabola, this MONTHLY, vol. 45, 1938, pp. 520–522.
28. Properties of points, lines, and circles associated with a point on an ellipse, this MONTHLY, vol. 48, 1941, pp. 435–443.

F. D. MURNAGHAN

29. Note on a paper by J. H. Weaver, this MONTHLY, vol. 32, 1925, pp. 37–41.

E. F. ALLEN

30. On a triangle inscribed in a rectangular hyperbola, this MONTHLY, vol. 48, 1941, pp. 675–681.

R. GOORMAGHTIGH

31. A theorem on a cyclic polygon, this MONTHLY, vol. 47, 1940, pp. 466–468.

32. A study of the quadrilateral inscribed in a circle, this MONTHLY, vol. 49, 1942, pp. 174–181.

33. Pairs of triangles inscribed in a circle, this MONTHLY, vol. 53, 1946, pp. 200–204.

34. Orthopolar and isopolar lines in the cyclic quadrilateral, this MONTHLY, vol. 54, 1947, pp. 211–214.

35. The Hervey point of the general n-line, this MONTHLY, vol. 54, 1947, pp. 327–331.

36. On the two-angle pole of a line to a triangle, this MONTHLY, vol. 55, 1948, pp. 71–75.

H. A. DOBELL

37. On the geometry of the triangle, this MONTHLY, vol. 39, 1932, pp. 71–85.

T. L. BENNETT

38. A theorem on hypocycloids, this MONTHLY, vol. 28, 1921, pp. 371–373.

A. M. PEISER

39. The Hagge circle of a triangle, this MONTHLY, vol. 49, 1942, pp. 524–527.

V. O. MCBRIEN

40. On some systems of orthopolar and Kantor lines of lines referred to a quadrangle, Catholic University of America Press, Washington, D. C., 1942.

O. J. RAMLER

41. The orthopole loci of some one-parameter systems of lines referred to a fixed triangle, this MONTHLY, vol. 37, 1930, pp. 130–136.

THE CARUS MONOGRAPHS

•

The Carus Monographs are expository presentations of the best thought and keenest researches in pure and applied mathematics, set forth in a manner comprehensible not only to teachers and students specializing in mathematics, but also to scientific workers in other fields. They are intended especially for the wide circle of thoughtful people who, having a moderate acquaintance with elementary mathematics wish to extend their knowledge without prolonged and critical study of the mathematical journals and treatises.

•

The complete list of Carus Monographs is:

No. 1. Calculus of Variations, by G. A. Bliss. 1925.

No. 2. Analytic Functions of a Complex Variable, by D. R. Curtiss. 1926.

No. 3. Mathematical Statistics, by H. L. Rietz. 1927.

No. 4. Projective Geometry, by J. W. Young. 1930.

No. 5. History of Mathematics in America before 1900, by D. E. Smith and Jekuthiel Ginsburg. 1934. (Out of print)

No. 6. Fourier Series and Orthogonal Polynomials, by Dunham Jackson. 1941.

No. 7. Vectors and Matrices, by C. C. MacDuffee. 1943.

No. 8. Rings and Ideals, by N. H. McCoy. 1948.

No. 9. The Theory of Algebraic Numbers, by Harry Pollard. 1950.

No. 10. The Arithmetic Theory of Quadratic Forms, by B. W. Jones. 1950.

No. 11. Irrational Numbers, by Ivan Niven. 1956.

The price of each monograph is $1.75 per copy to members of the Mathematical Association, one copy to each member, when ordered directly through the office of the Secretary of the Mathematical Association of America, University of Buffalo, Buffalo 14, New York.

Additional copies and copies for non-members of Monographs 1-8 are priced at $3.00 each, and must be purchased from the Open Court Publishing Co., LaSalle, Illinois.

In the case of Monographs 9-11, additional copies and copies for nonmembers must be purchased at $3.00 from John Wiley and Sons, 440 Fourth Ave., New York 16, N.Y.

THE SLAUGHT PAPERS

●

The Herbert Ellsworth Slaught Memorial Papers are a series of brief expository pamphlets published as supplements to the American Mathematical Monthly. The following four numbers have already been published:

1. *Fourier's Series; The Genesis and Evolution of a Theory* by R. E. Langer. v + 86 pages. Paper.

2. *Outline of the History of Mathematics (6th edition)* by R. C. Archibald. iv + 114 pages. Paper.

3. *Proceedings of the Symposium on Special Topics in Applied Mathematics.* iv + 73 pages. Paper.

4. *Contributions to Geometry.* iv + 75 pages. Paper.

●

Copies at one dollar each postpaid may be ordered from:

HARRY M. GEHMAN, *Secretary-Treasurer*
MATHEMATICAL ASSOCIATION OF AMERICA
University of Buffalo
Buffalo 14, New York